# Lecture # 8

**Review:** 1) Defn $\mathbb{C} = \{x + iy \mid x, y \text{ real}, i = \sqrt{-1}\}$.

2) FTA $\Rightarrow$ Every polyn. has a root over $\mathbb{C}$.

$\hookrightarrow$ $ax^2 + bx + c \longrightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

whenever $b^2 - 4ac < 0 \Rightarrow \mathbb{C}$-root.

**Defn:** A polynomial (over $\mathbb{C}$ or over $\mathbb{R}$) in the variables $z_1, \dots, z_n$ is a formal sum of terms (monomials) of the form $c \cdot z_1^{a_1} \cdots z_n^{a_n}$, $c$ is in $\mathbb{C}$. $a_i$ are pos. int.

$\hookrightarrow$ $n = 1$: $\Rightarrow$ 1-varable

Monomials are of the form

$c \cdot z^1, \; c \cdot z^2, \dots, c \cdot z^{77}$.

$\Rightarrow$ Polyn. as we've previous experienced.

$n = 2$: 2 variable, $x, y$

Mon $= c \cdot x^1 y^1, \; c x^1 y^2$

$c x^7, \; c \cdot y^8,$

$\Rightarrow$ Poly. i) $x^2 + y^2 - 1$

ii) $x^3 y^2 - 77x + 88y^2$

iii) $x^{2020} + 1944 y^6 x^6$

$n = 4$: $z_1, z_2, z_3, z_4$

$\hookrightarrow$ $z_1 z_2 - z_3 z_4^7$

$z_1^2$

$z_1^2 z_2^4 z_3^5 z_4 - z_2^2$.

**Nota:** We denote the set of all poly. in $n$ variables by $\mathbb{C}[z_1, \dots, z_n]$.

**Rmk:** $f(x) = x^4 - x^3 + x^2 - 1$ = real poly.

$\Rightarrow$ fcn $f : \mathbb{R} \longrightarrow \mathbb{R}$; $f(0) = -1$, $f(1) = 0$.

**Rmk:** Poly. in $n$-variables det. a fcn

$$f : \mathbb{C}^n = \mathbb{C} \times \dots \times \mathbb{C} \longrightarrow \mathbb{C}$$

$\hookrightarrow$ $f(x, y) = x^7 + i y^6 x^2$

$f(0,0) = 0$

$f(1,0) = 1 + 0 = 1$

$f(i, 1) = -i + i \cdot 1 \cdot (-1) = -2i$.

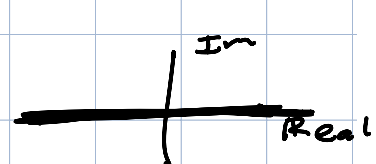
**Defn:** Given $f$ in $\mathbb{C}[z_1, \dots, z_n]$, we define

$$\mathbb{V}(f) = \{(w_1, \dots, w_n) \mid f(w_1, \dots, w_n) = 0\}.$$

$\hookrightarrow$ $f(z) = z^2 - 2z + 1 = (z-1)^2$

$\mathbb{V}(f) = \{ w \mid f(w) = 0 \}$

$= \{1\}$.

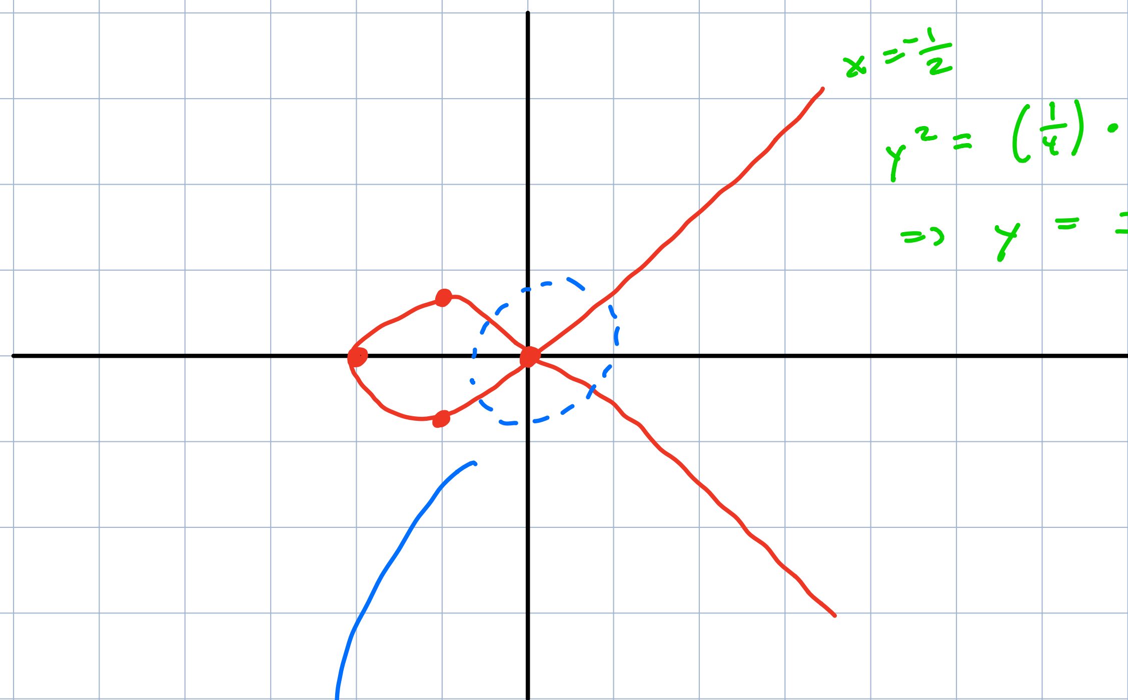


Ex: (Nodal cubic) $y^2 - x^2(x+1) = f(x,y)$

$$W(f) = \{ (x,y) \mid y^2 = x^2(x+1) \}.$$

$\hookrightarrow \quad \mathbb{R}^2 \subseteq \mathbb{C}^2$ via $(a,b) \longmapsto (a + 0\cdot i,\ b + 0 \cdot i)$

$\mathbb{R} \subseteq \mathbb{C}$ via $a \longmapsto a + 0 \cdot i.$

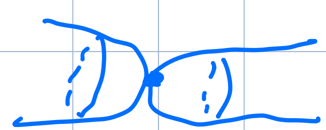


$x = -\frac{1}{2}$

$y^2 = \left(\frac{1}{4}\right) \cdot \frac{1}{2} = \frac{1}{8}$

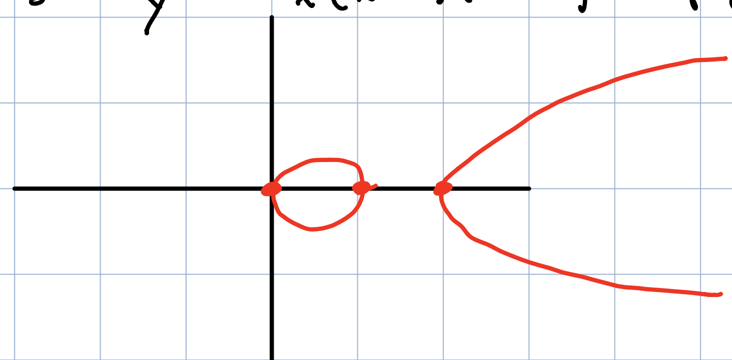
$\Rightarrow y = \pm \sqrt{1/8}$

in $\mathbb{C}^2$ it look like two disks kissing at a point



$\Rightarrow W(f)$ is <u>not</u> a surface.

Ex: Planar Cubic: $y^2 - x(x-1)(x-2) = f(x,y)$



↪ Looks like

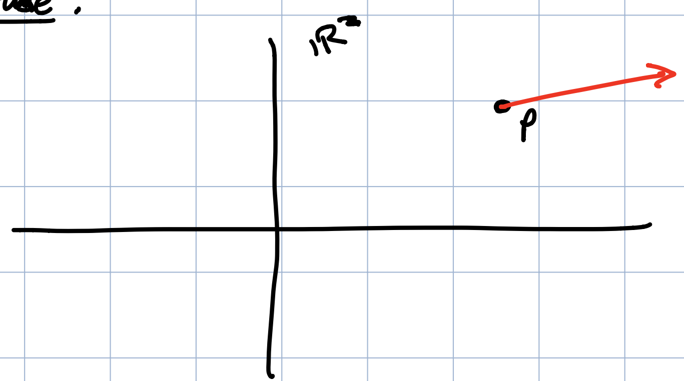
⟹ surface!

Ex: $y^2 = f(x)$ where $f(x)$ is a degree $n$ poly. w/ distinct zeos, then $V(f)$ = surface w/ $g$ holes where

$$\begin{cases} n = 2g+1 & \text{when } n = \text{odd} \\ n = 2g+2 & \text{when } n = \text{even.} \end{cases}$$

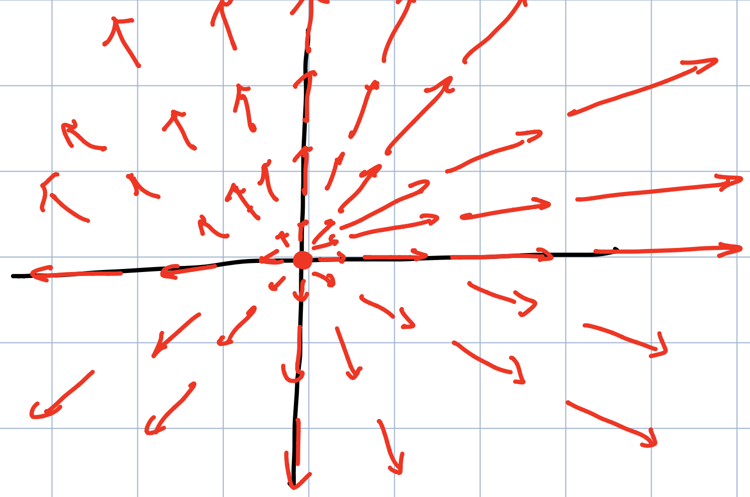
↪ $f(x) = x(x-1)(x-2)(x-3)(x-4)$.

Def: A vector at a point $p$ in $\mathbb{R}^2$ is a choice of <u>direction</u> and <u>magnitude</u>.



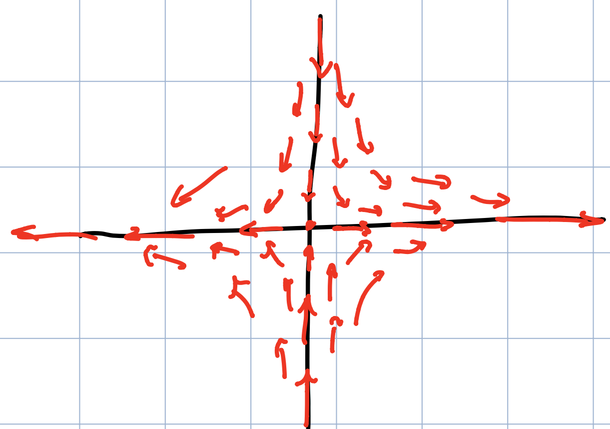

A vector field on $\mathbb{R}^2$ is a choice of vector at each point in $\mathbb{R}^2$ that satisfies: if $p, q$ close together then their vectors are also close.

Ex:



Ex:

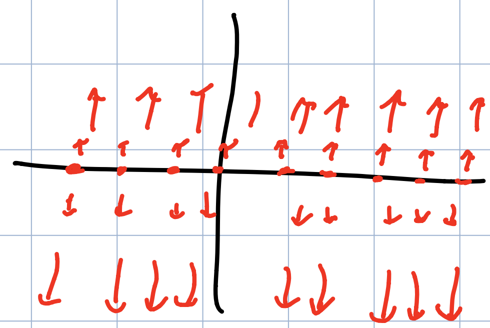
Ex:



Defn:
- If the magnitude of a vector is <u>zero</u> it is said to be a critical point of the vector field.
  - " "critical pts" ⟷ cowlicks when you comb hair.
- A vector field is non-degenerate if the critical

points are isolated.

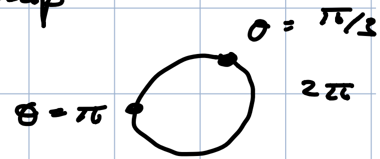
↪ Non-ex:



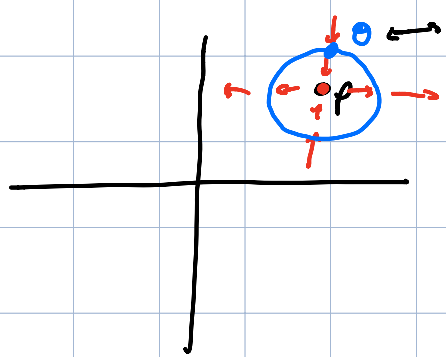
↪ "Most" vector fields are non-degenerate.

Defn: Given a crit. pt $p$, we have a map

$\gamma : S^1 \to$ Cylinder



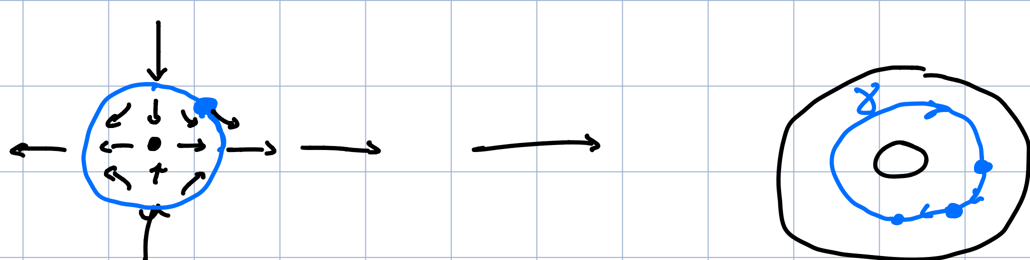

via sending $\theta$ in $S^1$



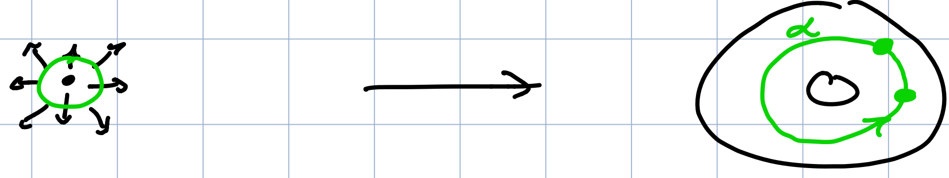

$\theta \mapsto$ vector $\longmapsto$ direction, magn on Cylinder

The index$(p) = \deg(\gamma)$.

Ex:




$\deg(\gamma) = -1$




$\deg(\alpha) = +1$.

Rmk: index $(p)$ is ind. of the choice of curve

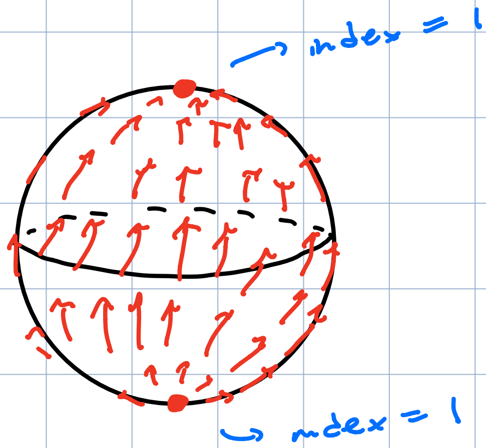
↪ why? $\forall$ two choices are homotopic

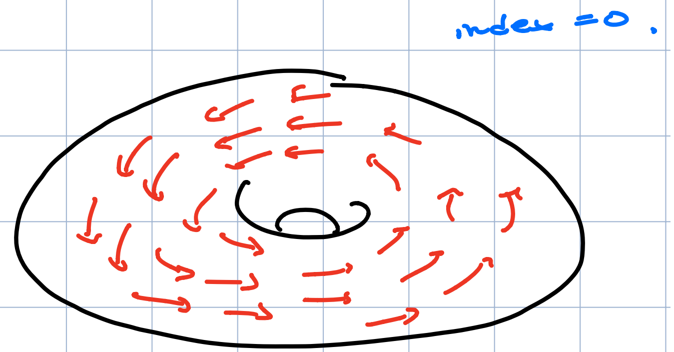
$\Rightarrow$ degrees agree.

Def: A vector at $p$ on a surface $\Sigma$ is a direction on the surf w/ a magnitude.

A vector field on a surf $\Sigma$ is a choice of vectors at every point.

Ex:




Rmk: the index of a critical point on the surf still makes sense.

Rmk:
$$\sum_{\substack{\text{index} \\ \text{crit.} \\ \text{on } S^2}} \text{index}(p) = 2 = \chi(S^2).$$

$$\sum_{\substack{\text{index} \\ \text{crit.} \\ \text{on } T^2}} \text{index}(p) = 0 = \chi(T^2).$$

Thm: (Poincare - Hopf Thm)

$\chi($oriented surface$)$ = sum of the indices of the crit points of <u>any</u> non-deg vector field.

Cor: You can't comb the hair on a ball w/out a cowlick.

Proof: Spse by way of contradiction that you could

$\Rightarrow$ vector field on $S^2$ w/ no critical points.

$\Rightarrow \chi(S^2) = 0$

But $\chi(S^2) = 2$

$\Rightarrow$ contradiction.