Lecture # 7

Outline:
1) Complex numbers
2) Complex Functions
3) Fund. Thm of Alg.
4) Alg. Sets

Defn: A real polynomial is a fcn $f: \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$$

where $a_0, \dots, a_n$ are real #s.

↳ $f(x) = x^2 - 2x + 1$

↳ $f(x) = 42 \cdot x^{77} + x^{66} - 3x^5 + 2$.

If $a_n \neq 0$, then we say that f has deg $= n$.

We say x_0 is a root of f if $f(x_0) = 0$.

Ex: $f(x) = x^2 + 1$.

$0 = f(x_0) = x_0^2 + 1$

$\Rightarrow -1 = x_0^2$

$\Rightarrow$ square of a real # is neg.

$\Rightarrow$ no such real # x_0 exists

Defn: A complex # is a formal sum of the form $z = x + iy$ where

i) x, y are real #

ii) i is a symbol that satisfies $i^2 = -1$

$\hookrightarrow$ $i = \sqrt{-1}$

Properties:

- Addition: $(x_0 + iy_0) + (x_1 + iy_1) =$
$$= (x_0 + x_1) + i(y_0 + y_1).$$

- Multiple: $(x_0 + iy_0)(x_1 + iy_1)$
$$= x_0 x_1 + i x_0 y_1 + i x_1 y_0 + \underbrace{i^2}_{=-1} y_0 y_1$$
$$= (x_0 x_1 - y_0 y_1) + (x_0 y_1 + x_1 y_0) i$$

- Complex Conjugate: $z = x + iy$
$$\overline{z} = \text{cpx conj of } z = x - iy.$$

Fact: $z = x + iy$, then $x^2 + y^2 = z \cdot \overline{z}$.

Proof: $(x + iy)(x - iy) = x^2 - ixy + ixy - i^2 y^2$
$$= x^2 + y^2.$$

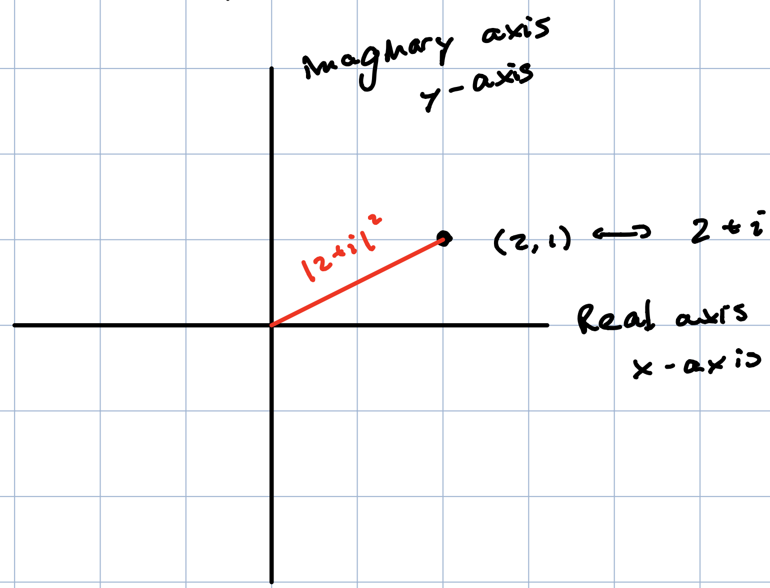
$\hookrightarrow$ real #!

Defn: · Division: $\dfrac{x_0 + iy_0}{x_1 + iy_1} \cdot \dfrac{x_1 - iy_1}{x_1 - iy_1} = \dfrac{z_0 \cdot \overline{z}_1}{z_1 \cdot \overline{z}_1}$

Defn: $|z|^2 = z \cdot \overline{z} = x^2 + y^2$

Remark: We can identify $\mathbb{R}^2$ w/ $\mathbb{C}$ = cpx numbers.

$(x, y) \longmapsto x + iy$ in $\mathbb{C}$.



Defn: A fcn of $\mathbb{C}$-numbers is an assignment of each z in $\mathbb{C}$ to some $f(z)$ in $\mathbb{C}$.

$f : \mathbb{C} \to \mathbb{C}$.

↳ $f(z) = z^2$.

↳ $f(1) = 1$

$f(i) = i^2 = -1$

$f(-1) = (-1)^2 = 1$

$f(2+i) = 3 + 4i$

Defn: A complex polynomial is a fcn $f : \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z^1 + a_0$$

where $a_0, \dots, a_n$ are cpx #s.

$\Leftrightarrow \quad f(z) = iz^2 + (2+i)z - 42.$

Defn: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Ex: $1 = 0$;

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots \quad 3$$

$$(1-1) + (1-1) + (1-1) + (1-1) + \cdots = 0 \quad 1$$

$$1 + (-1+1) + (-1+1) + (-1+1) + \cdots = 1 \quad 2$$

$$1 = 2 = 3$$

Rmk:

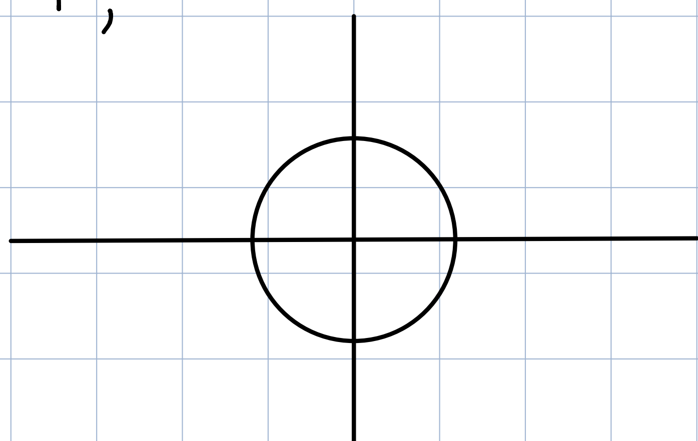
$$\underbrace{1 + z + \frac{z^2}{2} + \cdots + \frac{z^N}{N!}}_{\text{approximation.}} + \underbrace{\text{other stuff}}_{\text{error term}}$$

Theorem: $e^{i\theta} = \cos(\theta) + i\sin(\theta).$ (θ is a real #)

Rmk: $\mathbb{R}^2 \longleftrightarrow \mathbb{C}$

$\{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ is the set of $\mathbb{C}$ st

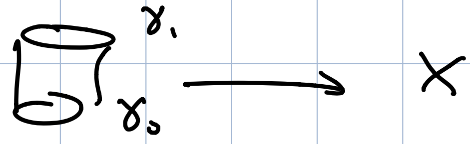
$|z|^2 = 1$,



Thm: (Fund. Thm of Alg. - Gauss) Every poly has a root over the complex numbers.

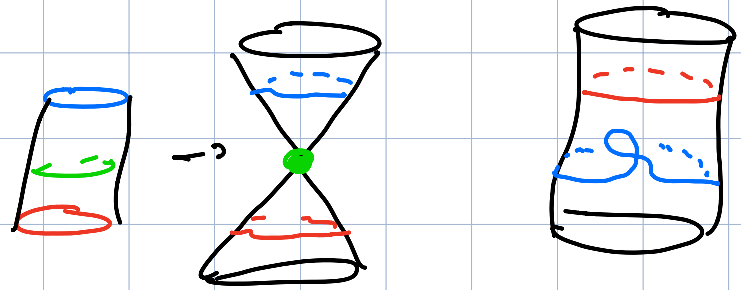
Defn: 1) Two closed curves $\gamma_0 : S^1 \to X$, $\gamma_1 : S^1 \to X$ are homotopic if

a) There exists a map of a cylinder



st top circle is mapped via γ_1
bottom " " " " γ_0

$\rightsquigarrow$ $\gamma_i : S^1 \to$ Cylinder



b) γ_0, γ_1 are homotopic if there exist a map $H : S^1 \times [0,1] \to X$ st

$$H(s,0) = \gamma_0(s)$$
$$H(s,1) = \gamma_1(s)$$

Defn: $\gamma : S^1 \to$ Cylinder, then $\deg(\gamma)$ = signed # of times γ wraps around the cylinder.

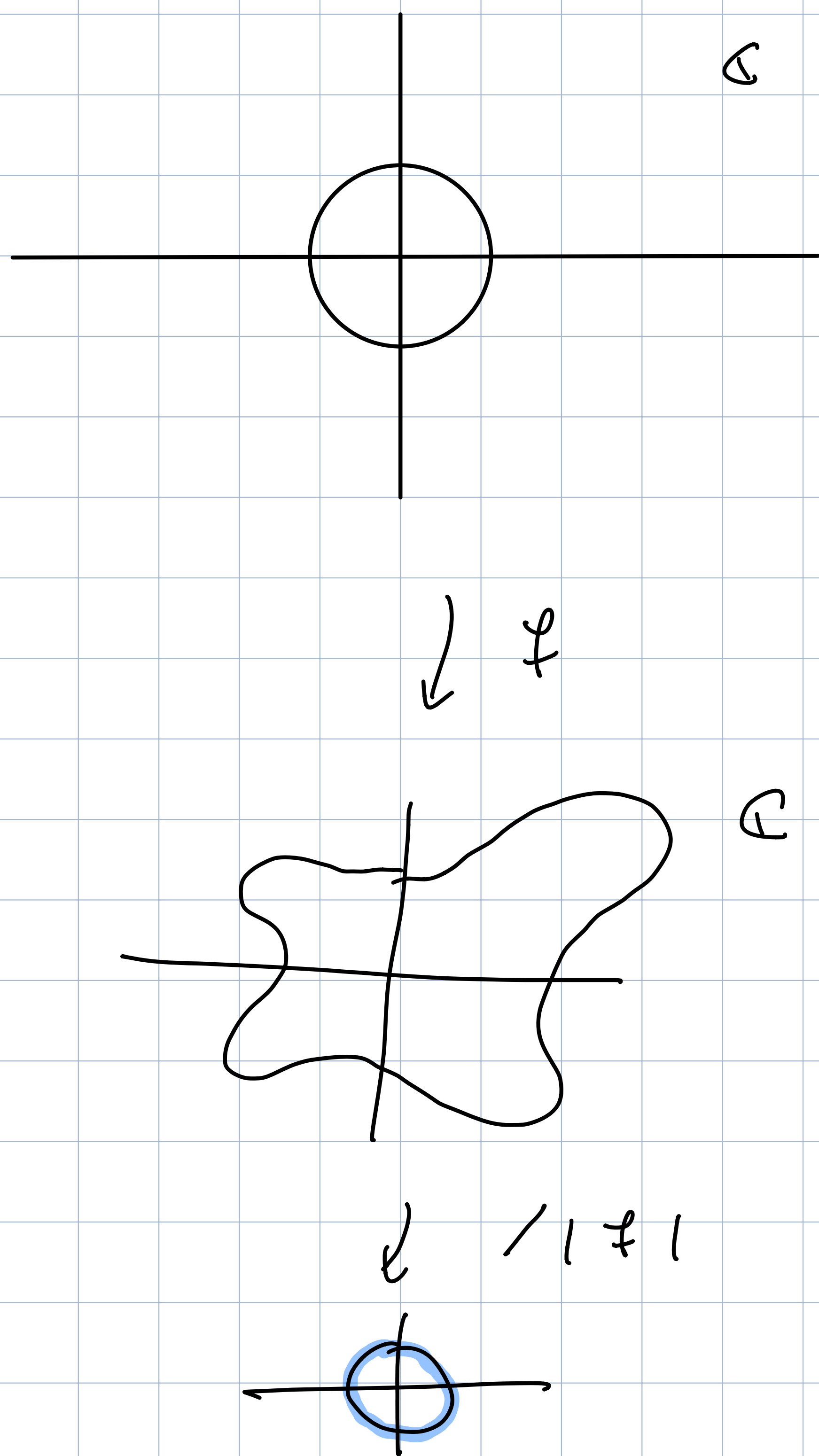
Thm: $\deg(\gamma_0) = \deg(\gamma_1)$ iff γ_0 is homotopic to γ_1

Proof: Spse by way of contradiction that

$$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

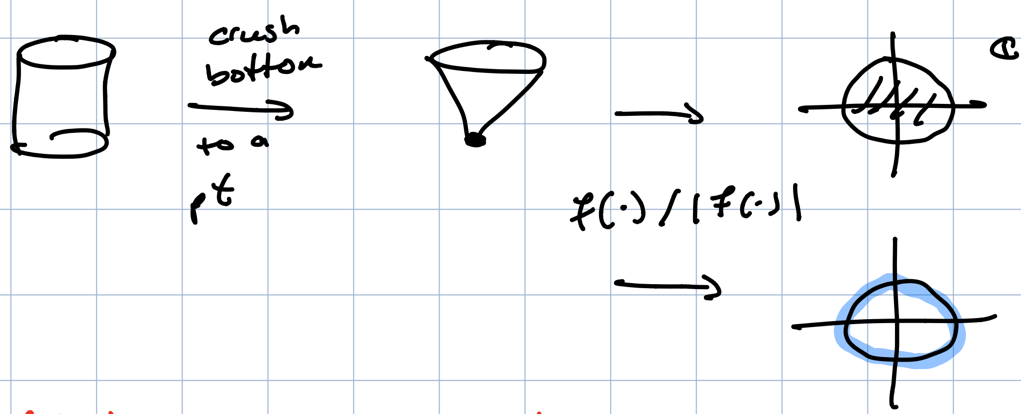
has no roots, ie, $f(z_0) \neq 0$ for all z_0 in $\mathbb{C}$.

Define $\gamma(\theta) = f(e^{i\theta}) / |f(e^{i\theta})|$



γ as a curve into cylinder by thickening unit circle in $\mathbb{C}$.

Claim: $\deg(\gamma) = 0$.



Note $f(z_0) \neq 0 \Rightarrow |f(\cdot)| \neq 0$.

deg (bottom curve on this cylinder) $= 0$

$\hookrightarrow$ map is to a pt $\Rightarrow$ no wrapping.

Define $\gamma_s = s^n f(e^{i\theta}/s) \,/\, |s^n f(e^{i\theta}/s)|$ $\quad 0 \le s \le 1$

$s^n f(z/s)$

$= s^n \left(z^n/s^n + a_{n-1} z^{n-1}/s^{n-1} + \dots + a_1 z/s + a_0 \right)$

$= z^n + a_{n-1} z^{n-1} s + \dots + a_1 z s^{n-1} + s^n a_0 = (\star)$

$s = 1, \quad (\star) = f(z)$

$s = 0, \quad (\star) = z^n$.

$\gamma_1 = \gamma$

$\gamma_0(\theta) = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$.

So as θ goes from 0 to 2π, γ_0 wraps n times around the cylinder.

$\deg(\gamma_0) = n = \deg(\gamma_1) = \deg(\gamma) = 0$.

So when $n > 0 \Rightarrow$ contradiction

$\Rightarrow$ assump was wrong, ie, f had a root.