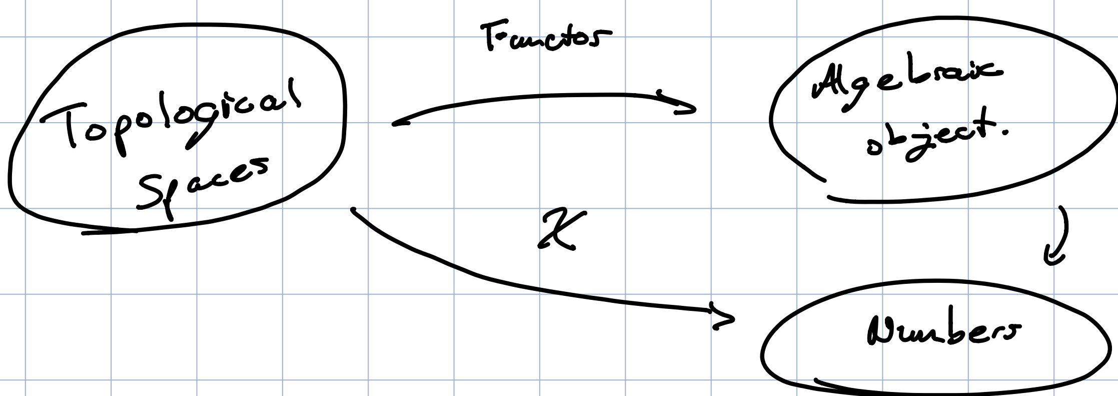
Lecture # 6

Outline:
1) Homotopy
2) Degree
3) Applications
   ↳ Brouwer's Fixed Point Theorem
   ↳ Fundamental Theorem of Algebra

Remark: Category theory.

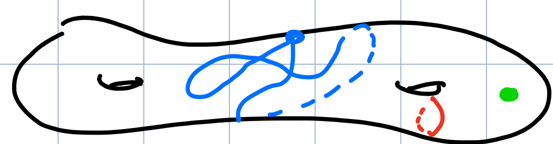
- Categories = collection of things and maps between things.
- Functors = maps between categories



Defn:
- A closed curve is a cont. map

$$\gamma : S^1 \longrightarrow X \text{ - top space.}$$

↳

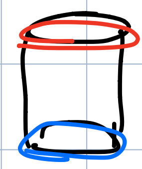


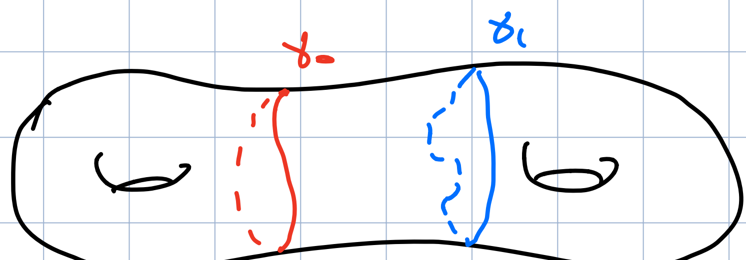
- Two closed curves $\gamma_0 : S^1 \to X$ and $\gamma_1 : S^1 \to X$ are homotopic if there exist a map

$f:$ Cylinder $\longrightarrow X$

st $f(\text{top circle}) = \gamma_0$

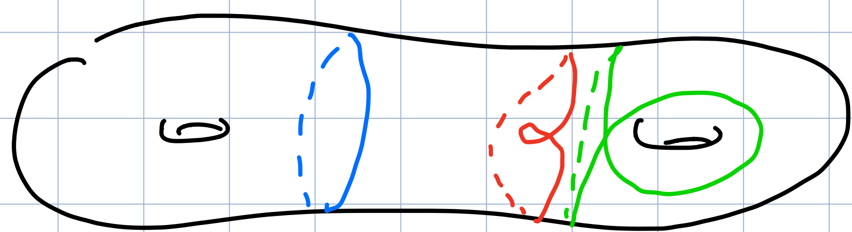
$f(\text{bottom circle}) = \gamma_1$





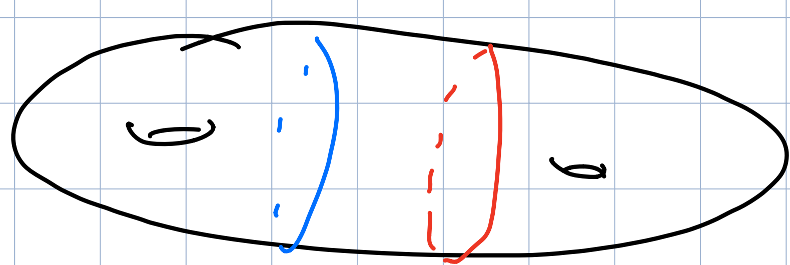
$\Rightarrow$ deform the γ_1 into γ_0.

Not hptic.

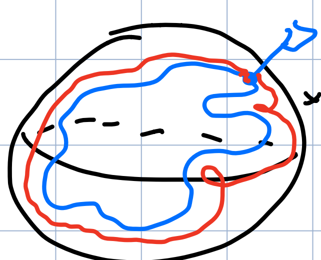


Are hptic

not hptic



Picture:

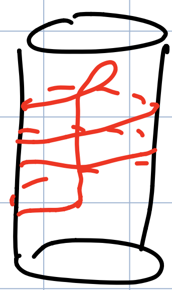


deformation to give hpty are operation an ant could do w/ a rope.

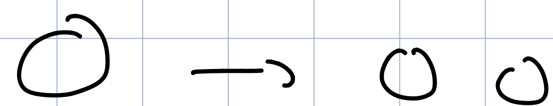
Defn: Let $\gamma : S^1 \to \text{Cylinder} = S^1 \times [0,1]$. The degree of γ, $\deg(\gamma)$, is the signed # of times γ fully wraps around the cylinder.

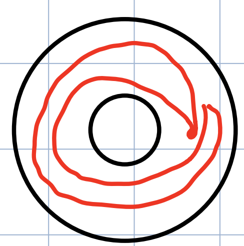
- sign: ↺ = pos.

  ↻ = neg.

- 

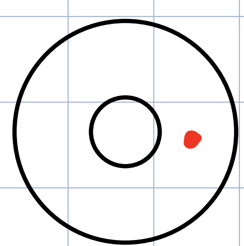
  wraps 2x $\Rightarrow$ degree = 2.

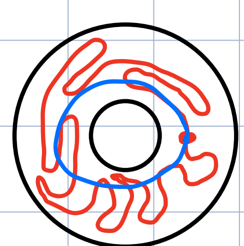
- 

  wraps 1 pos. $\Rightarrow$ deg = 0.

  1 neg

- 

  deg = 0.

- 

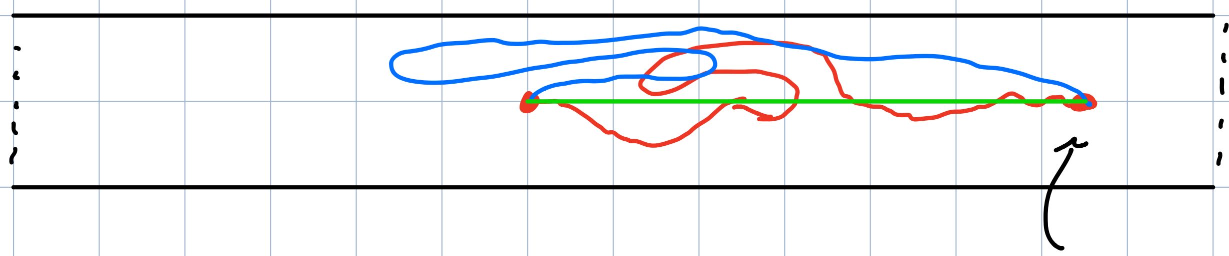
  $\deg(\xi) = 1$

Theorem: Two curves $\gamma_0 : S^1 \to S^1 \times I$ and $\gamma_1 : S^1 \to S^1 \times I$ are homotopic if and only if $\deg(\gamma_0) = \deg(\gamma_1)$.

"Proof:" • Unwrap curves in $\mathbb{R} \times I$

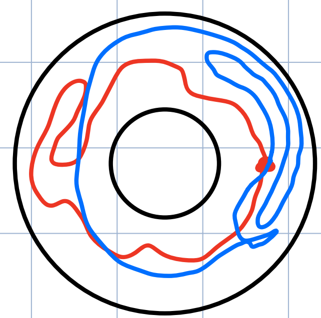
• $\mathbb{R} \to S^1$

$\mathbb{R} \times I$



↓ wrap around

end point is the degree of the curve.



upstairs we pull the cord tight.

$\Rightarrow$ new curve of the same degree that is homotopic to red, blue.

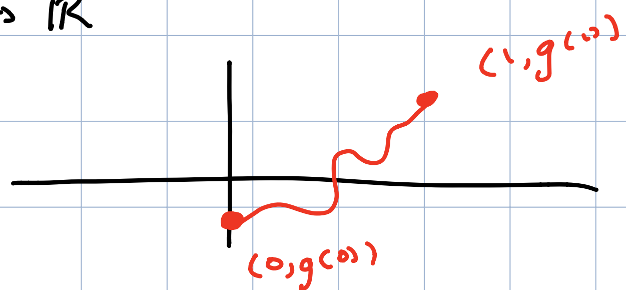
Push the deformation downstairs.

Thm: (1-dim) If $f : [0,1] \to [0,1]$ continuous, then there exists x in $[0,1]$ st $f(x) = x$.

Proof: $g(x) = x - f(x)$.

$$g(0) = 0 - f(0) \leq 0 \leq 1 - f(1) = g(1).$$

$g : [0,1] \to \mathbb{R}$



Intermediate value Thm.

$\Rightarrow$ there exist x st $g(x) = 0$

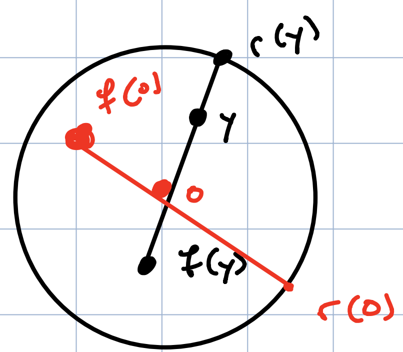
$\Rightarrow$ $x - f(x) = g(x) = 0$

$\Rightarrow$ $f(x) = x$. $\square$

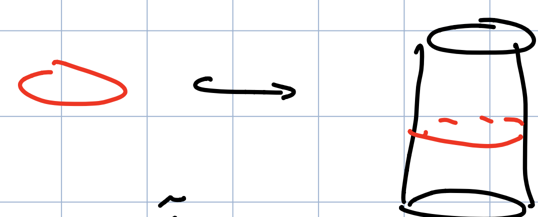
Thm: (Brouwer's Fixed Point Thm - 2-dim) If f is a continuous map from the 2-dim disk to itself, then there exist x in the disk st $f(x) = x$.

Proof: Spse by way of contradiction that f has <u>no</u> fixed points, i.e., $f(x) \neq x$ for all x.

We define a map of r: Disk $\longrightarrow S^1$ as follows:



$\tilde{r}$: Disk $\longrightarrow$ Cylinder by including the "equator"

$\tilde{r}$: Disk $\longrightarrow$ 

$\gamma_0 : S^1 \to$ boundary of Disk. $\xrightarrow{\tilde{r}}$ cylinder.

$\gamma_1 : S^1 \to$ origin $\xrightarrow{\tilde{r}}$ cylinder

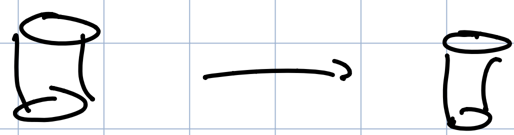
- $\deg(\gamma_1) = 0$

  $\deg(\gamma_0) = \pm 1 \neq 0$

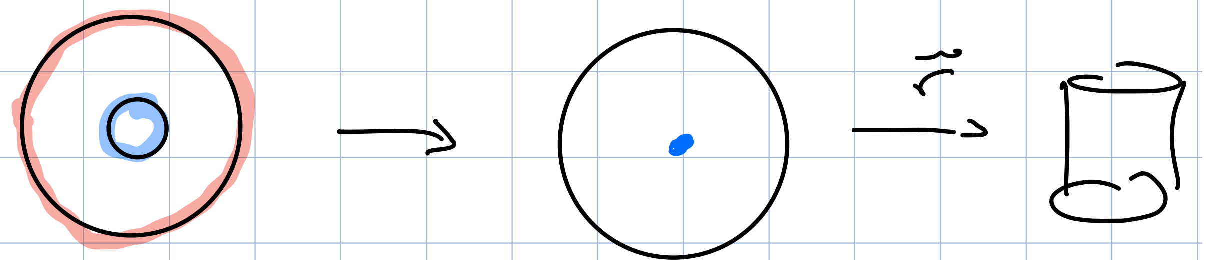
Claim $\deg(\gamma_1) = \deg(\gamma_0)$

Thm $\Rightarrow$ suff to show γ_1 is homotopic to γ_0

$\Rightarrow$ cylinder



st top is γ_1 and bottom is γ_0.



top is just γ_0, bottom is γ_1

$\Rightarrow$ γ_0 is homotopic to γ_1

$\Rightarrow$ $0 = \deg(\gamma_1) = \deg(\gamma_0) = \pm 1$.

$\Rightarrow$ Contradiction

$\Rightarrow$ original assumption is false.

$\Rightarrow$ f has a fixed point.