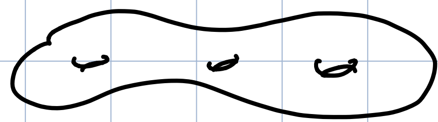
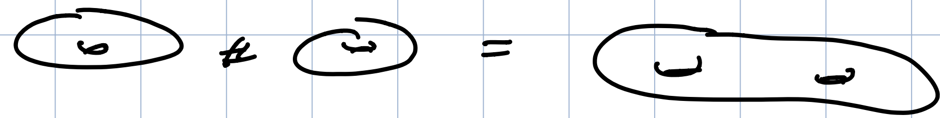
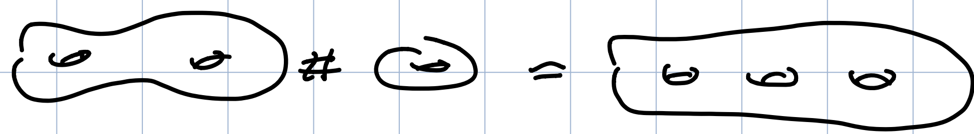
Lecture # 5

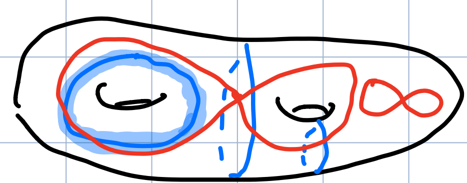
Thm: (Classification of surfaces) X = compact, orientable surface then $X \cong T^2 \# \cdots \# T^2$ or $X = S^2$.

↪  $\cong T^2 \# T^2 \# T^2$

↪ 



Defn: A simple closed curve (scc) on surf X is a loop on X that doesn't intersect itself

↪ 

= non-ex

= examples.

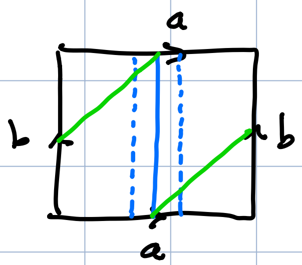
Defn: A scc is 2-sided if a small thickening of the curve is an annulus / cylinder (2 boundary comps)

" " " 1-sided " " " " " " "

" " Mobius band (1 boundary component)

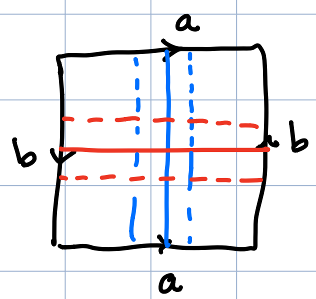
Ex:

i)



⇝ 2-sided curve

ii)



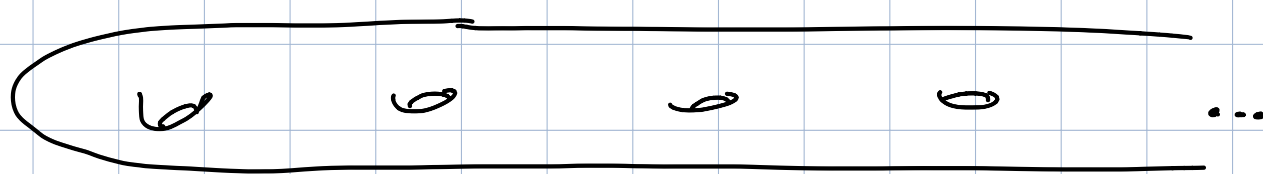
= 2-sided curve

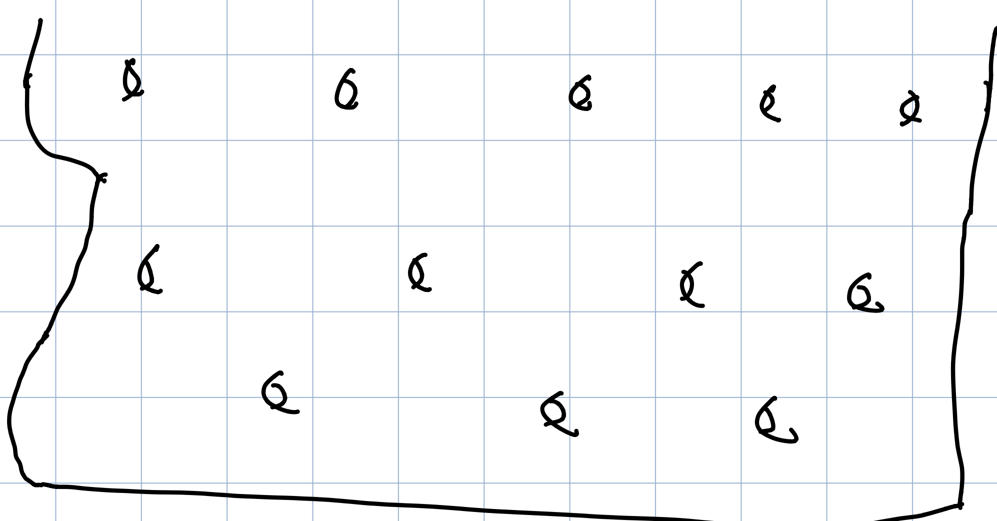
= 1-sided curve

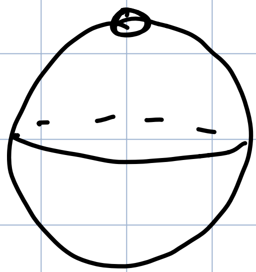
Defn: A surf is orientable if it has no 1-sided curves.

Defn: A surf is compact if a poly cpx associated to it has a finite # of polygons

Ex:

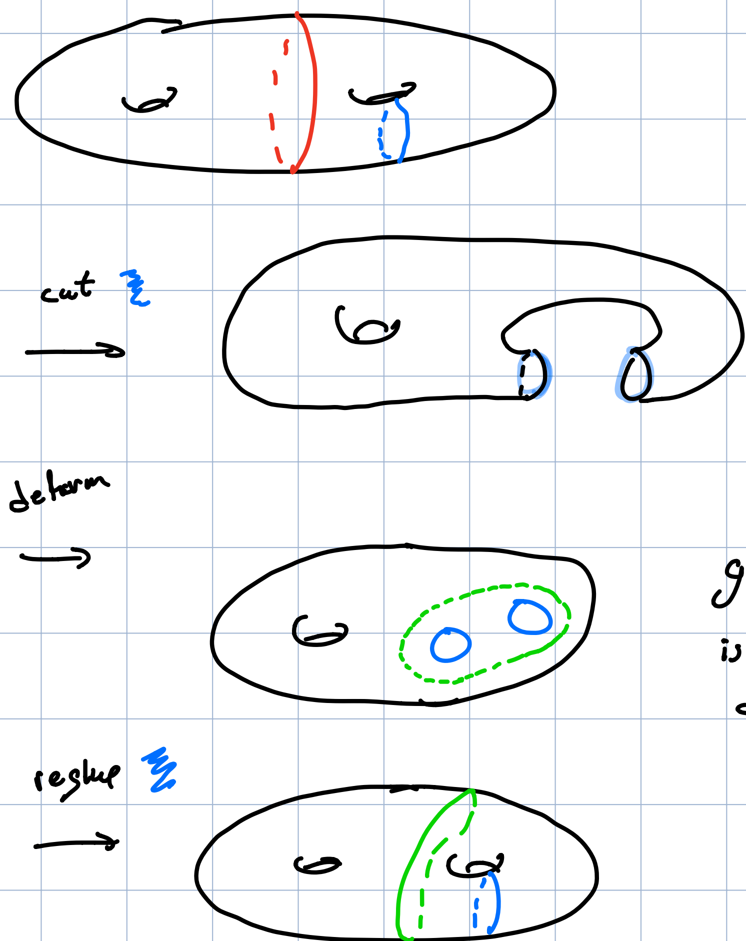






Rmk: If γ is a 2-sided curve in X that does not sep. X, then $X = T^2 \# Y$, where Y is some other surf.

cut along it and X w/ the cut is still conn.



green w/ blue glued is a T^2 w/ a boundary comp

$\Rightarrow$ X is realized as a connect sum w/ T^2 along the green curve

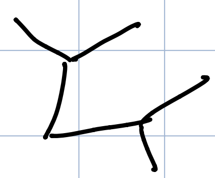
Rmk: $X \# S^2 = X$

Lemma: Γ = conn graph, then

$$\chi(\Gamma) \leq 1$$

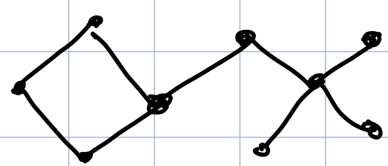
w/ $\chi(\Gamma) = 1$ if and only if Γ = tree.

Ex: i)



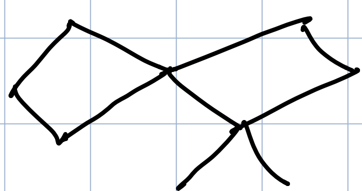
$7 - 6 = 1$ ✓

ii)



$9 - 9 = 0$

iii)



$9 - 10 = -1$

Rmk: Each loop contributes -1 to the χ.

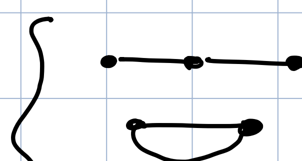
Proof: Build Γ up (in a conn manner) in a seq.

$\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_N$.

Compute $\chi(\Gamma_i)$ inductively

$\Gamma_1 =$  $\Rightarrow \chi(\Gamma_1) = 1$

$\Gamma_2 =$



$\Rightarrow \chi(\Gamma_2) = 3 - 2 = 1$

$\Rightarrow \chi(\Gamma_2) = 2 - 2 = 0 \leq 1$

Γ_{i+1} from Γ_i, we ① either add 1 new edge and 1 new ver. or ② just add 1 new edge.

① $\Rightarrow$ $\chi(\Gamma_{i+1}) = V(\Gamma_{i+1}) - E(\Gamma_{i+1})$

$$= V(\Gamma_i) + 1 - (E(\Gamma_i) + 1)$$
$$= V(\Gamma_i) + E(\Gamma_i)$$
$$\leq 1$$

② $\Rightarrow$ $\chi(\Gamma_{i+1}) = \chi(\Gamma_i) - 1$

Note if we just do ①, then we create no loops $\Rightarrow$ Γ is a tree but we don't drop $\chi(\Gamma)$ from 1 w/ this set-up.

Thm: (2-dim'l Poincaré Conjecture) If X = surf w/ $\chi(X) = 2$, then $X = S^2$.

Cor: X surf, then $\chi(X) \leq 2$

Proof: Fix some poly cpx for X

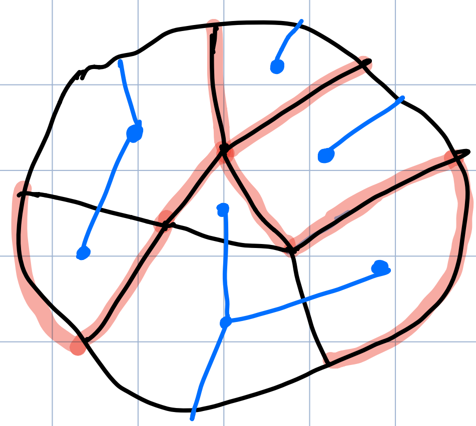
Note edges form a graph in X, say Γ

Pick spanning tree for Γ, say T.

Defn another graph Γ' on X as follows

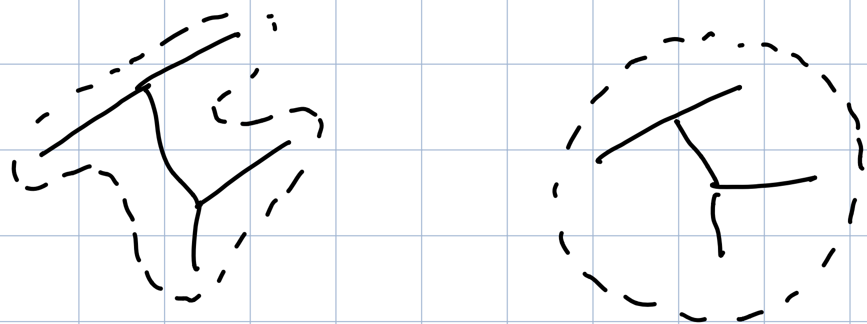
i) at each face we drop a vertex

ii) if two faces are conn via edge $\Rightarrow$ conn vertices.



Claim Γ' is connected

↳ $T \subset X$ thickened up is a disk.



$\Rightarrow$ X - disk is connected but Γ' lives on X - disk $\Rightarrow$ Γ' is connected itself.

$$\begin{aligned}\chi(X) &= V - E + F \\ &= \underline{V(T)} - (\underline{E(T)} + \underline{E(\Gamma')}) + \underline{V(\Gamma')} \\ &= \chi(T) + \chi(\Gamma')\end{aligned}$$

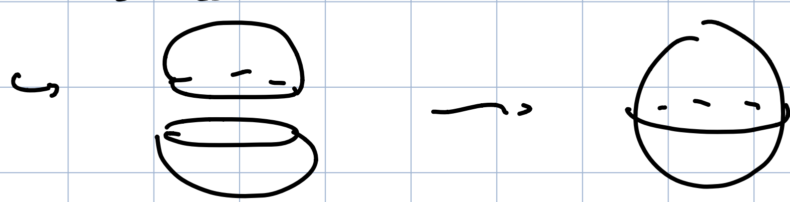
$$2 = \chi(X) = \chi(T) + \chi(\Gamma') = 1 + \chi(\Gamma')$$

$\Rightarrow$ Γ' = tree.

Slowly thicken T and Γ' to fill up X

But thick(T), thick(Γ') are both homeo to disks.

$\Rightarrow$ fill up realizes X as two disks glued along their boundaries.



$\square$

Proof: Let T, Γ' be as before.

$\Rightarrow \chi(X) = \chi(T) + \chi(\Gamma') = 1 + \chi(\Gamma')$.

If $\chi(X) = 2 \Rightarrow$ done $\Rightarrow X = S^2$ ☺

If $\chi(X) < 2 \Rightarrow 2 > 1 + \chi(\Gamma')$

$\Rightarrow 1 > \chi(\Gamma')$

$\Rightarrow \Gamma'$ is not a tree

$\Rightarrow \Gamma'$ has a loop.

Call this loop γ.

X is orientable $\Rightarrow \gamma$ = 2-sided curve!

$\Rightarrow$ using γ we realize X as a connect sum

$$X = T^2 \# X'$$

Note,

$$\chi(X) = \chi(T^2) + \chi(X') - 2$$
$$= 0 + \chi(X') - 2$$

$\Rightarrow \chi(X') > \chi(X)$.

$$X = T^2 \# X' = T^2 \# T^2 \# X''$$

Eventually X'', X''' will have $\chi \geq 2$

$\Rightarrow$ eventually we get

$$X = T^2 \# \cdots \# T^2 \# S^2$$
$$= T^2 \# \cdots \# T^2.$$

Scratch: