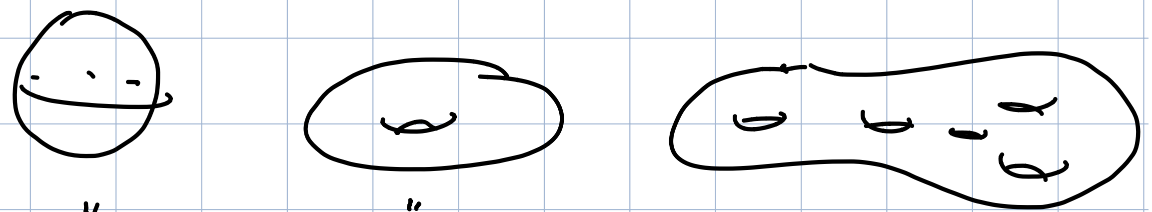
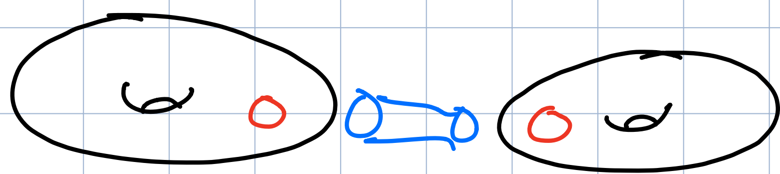
Lecture # 4

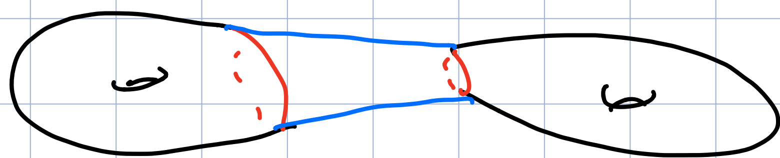
Recall: 1) Surfaces = locally looked like pieces of paper



$\shortparallel$ S^2 $\qquad$ $\shortparallel$ T^2

2) Connect sums





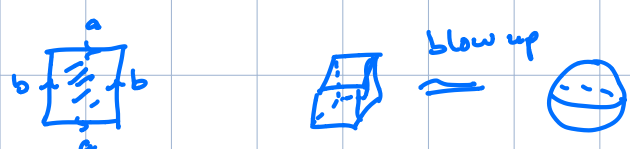
↳ ∞ many diff surfaces.

3) Euler characteristic

↳ $\chi(\text{poly. cpx}) = V - E + F$

↳ $\chi(\text{surface}) = \chi(\text{any poly. cpx that homeo to surf})$

↳ a way of breaking up surface into polygons.

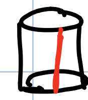
↳ 

↳ $\chi(T^2) = 0$ $\qquad$ $\chi(T^2 \# T^2) = -2$

$\chi(S^2) = 2$ $\qquad$ $\chi(g \text{ holes}) = 2 - 2g$

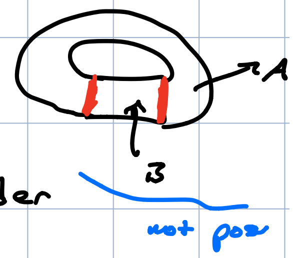
Defn: Geographic cpx is a poly cpx

i) Faces do <u>not</u> meet themselves



↳ no country has border w/ self

ii) Faces share unique edges



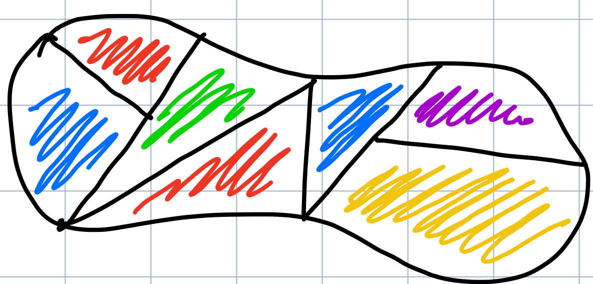
↳ pair of countries have unique border

↳ two countries either border along a single edge or are separate

iii) at each vertex at least 3 faces meet

↳ vertices are where 3 countries come together.

Defn: A legal coloring of a geog. cpx is coloring of faces st no two adj. faces have the same color



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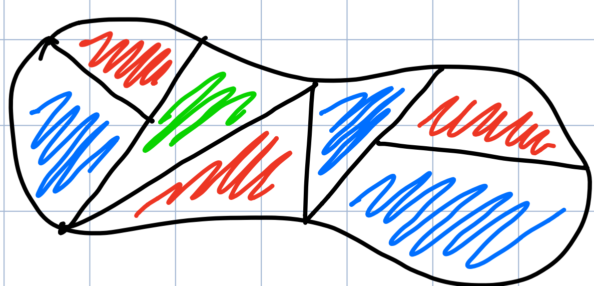


= not OK

Defn: The coloring # of a geog cpx is the min # of colors needed to prod. a legal coloring.

Defn: The coloring # of a surface is the min # of colors needed to " " " " of <u>any</u> geog cpx homeo to surface

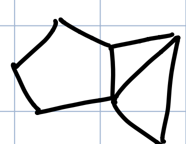
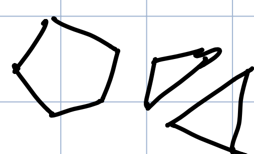
↳ pick any map and you can color it w/ this # of colors

Notn: $N(K = \text{geog})$ = legal coloring #

$N(X = \text{surf})$ = " " "

Thm:
$$N(X) \leq \frac{7 + \sqrt{49 - 24\chi(X)}}{2}$$

Notn:
- K = geo cpx on X w/ the following prop:
  - i) $N(K) = N(X)$
  - ii) K' is another geo cpx on X st $N(K') = N(X)$ then $F(K) \leq F(K')$
- $\tilde{K}$ = poly cpx that is the ~~pre~~-un-gluing of K

 = K  = $\tilde{K}$

rip apart edges

↳ disj coll of poly gons.

$$\frac{5+3+3}{3} = \frac{11}{3} \qquad \frac{3+3+5}{3} = \frac{11}{3}$$

Lemma: $(N(K)-1)F(K) \leq 2E(K)$

$\iff N(K)-1 \leq 2E/F.$

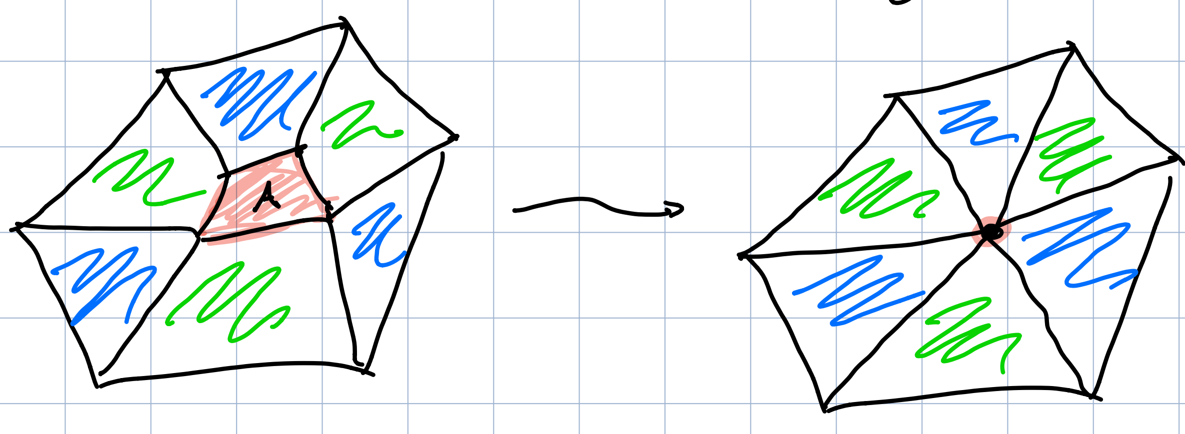
Proof: i) Average # of edges per face is $2E/F$.

$$\text{Ave} = E(\tilde{K})/F(\tilde{K})$$
$$= E(\tilde{K})/F(K)$$
$$= 2E(K)/F(K)$$

ii) To finish proof, we need to show that each face has at least $N(K)-1$ edges.

$\Rightarrow \text{Ave} \geq (N(K)-1)/F(K)$

Assume by way of contradiction that there is a face w/ less than $N(K)-1$ edges.



Modify K to prod. a new geo cpx. Shrink face to a vertex.

Assp on geo cpx $\Rightarrow$ this shrunk guy is in fact geo cpx. $= K'$

$N(K') \leq N(K)$ $\quad N = N(K)$

But $F(K') < F(K)$

So by con of K, $N(K') < N(K)$.

Coloring of K' gives a coloring of K via $N(K)-1$

colors.

Color K' w/ $N(K)-1$ colors, but since A has $N-2$ surr faces, we can color K w/ $N(K)-1$ colors, contradiction.

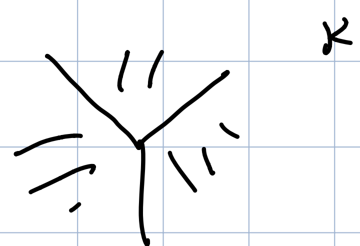
$\Rightarrow$ every face has at least $N-1$ edges.

Lemma: $(N(K)-1)F(K) \le 2E(K)$

$$\iff N(K)-1 \le 2E/F.$$

Lemma: $3V(K) \le 2E(K)$

Proof: Use $\tilde{K}$ = un-glued cpx



$$3V(K) \le V(\tilde{K}) = E(\tilde{K}) \le 2E(K)$$

$\hookrightarrow$ collect of polygon

Lemma: $N(K)-1 \le 6 - 6\chi(X)/F(K)$

Proof: $(N-1)F \le 2E \le 6E-6V$

$$= -(6V-6E+6F)+6F$$
$$= -6(V-E+F)+6F$$
$$= -6\chi(X)+6F$$

$3V \le 2E$
$6V \le 4E$
$0 \le 4E-6V$
$2E \le 6E-6V$

$\square$

Proof: Of coloring thm

i) $\chi(X) = 1$

$$N(X) \leq \frac{7 + \sqrt{49 - 24}}{2} = \frac{7+5}{2} = 6$$

By lem,

$$N - 1 \leq 6 - 6/F(K) \leq 5$$

$$\Rightarrow N(X) \leq 6$$

✓

ii) $\chi(X) \leq 0$

not poss, $N(K) \leq F(K)$

$$\hookrightarrow N - 1 \leq 6 - 6\chi(X)/F(K) \leq 6 - 6\chi(X)/N$$

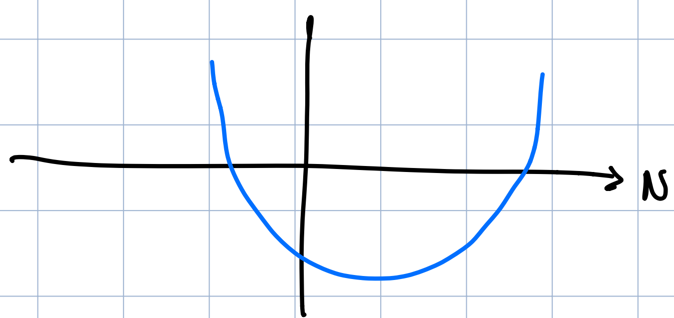
Mult by N

$$N(N-1) \leq 6N - 6\chi(X)$$

$$N^2 - N \leq 6N - 6\chi(X)$$

$$N^2 - 7N + 6\chi(X) \leq 0$$

↳ poly in N



↳ zero of this poly gives an optimal solution to problem

Quad form

$$N \leq \frac{7 + \sqrt{49 - 24\chi(X)}}{2}$$

□

Ex: $X = T^2$, $\chi(T^2) = 0$

$$\Rightarrow N(T^2) \leq \frac{7 + \sqrt{49 - 0}}{2} \leq 7$$

Exer: Find a geo cpx on T^2 that requires 7 colors to be legally colored

$\Rightarrow N(T^2) = 7$

Rmk: Planar dgms are never geo cpx (b/c faces edges w/ itself)