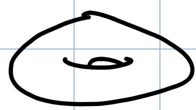
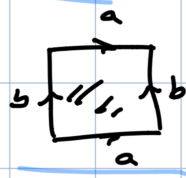
Lecture #3

Defn: A surface is a top. space that loc. looks like a piece of paper (loc. 2-dim.)

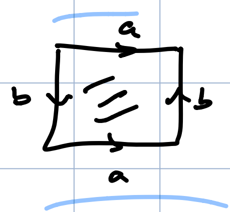
↳  $= S^2 =$ 

Combinatorial models for surfaces
↳ planar dgm

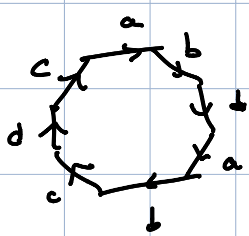
↳  $= T^2 =$ 

↳ 

$= P^2 =$ real projective plane

↳ 

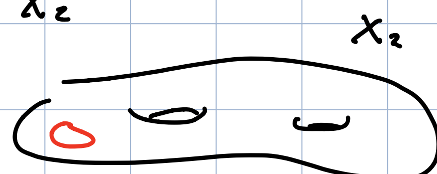
$= K =$ Klein bottle

↳ 

$\simeq$?

Defn: A connect sum of two surfaces X_1, X_2 is the surf X_3 obtained as follows

1) remove disks from X_1 and X_2



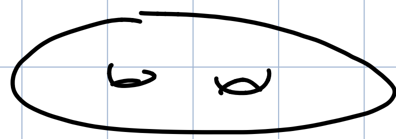
2) Glue resulting boundary comp. together

 $= X_3$

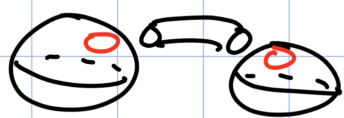
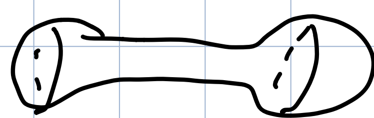
we denote this by $X_3 = X_1 \# X_2$

$\rightsquigarrow$ this construction is ind of the disks that we remove.

Ex: i) $T^2 \# T^2$ = intertube w/ 2 holes

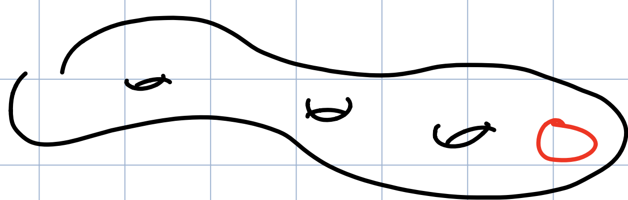
$=$ 

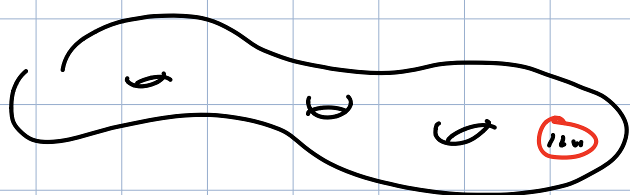
ii) $S^2 \# S^2 = S^2$

 $=$ 

$=$ 

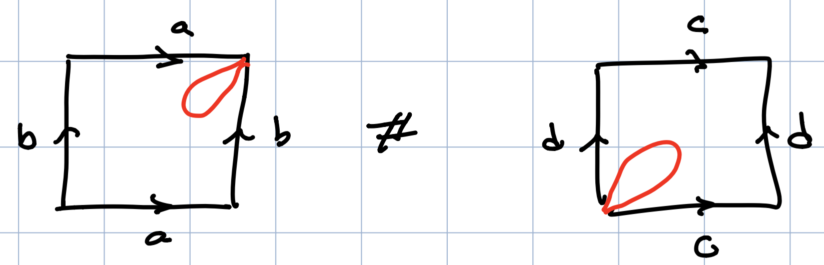
iii) $X \# S^2 = X$

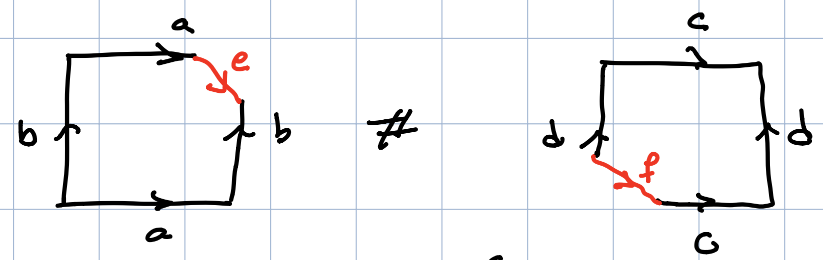
 $\#$ 

$=$ 

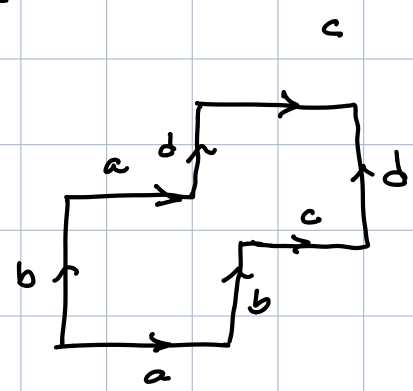
Rmk: Connect sum can be realized on level of planar dgms

Ex: $T^2 \# T^2$

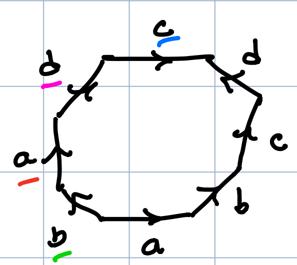




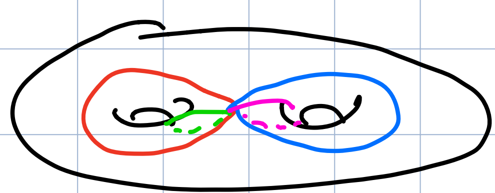
=



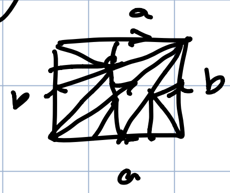
=



=

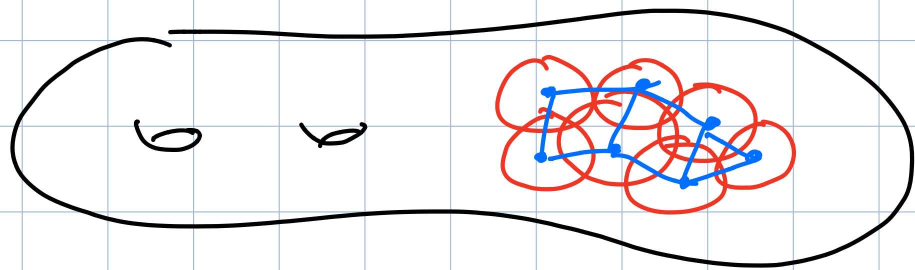


Rmk: Each surf has ∞ # of poly cpxes that are homeo to surface. We call these poly structures/ triangulations.

$\hookrightarrow$ 

Thm: Every surface admits a decomp into a poly cpx.

Proof:



$\square$

Defn: The Euler characteristic of a poly cpx K is

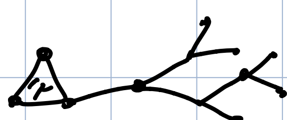
$$\chi(K) = V(K) - E(K) + F(K)$$

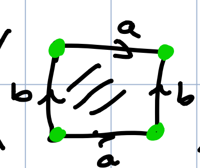
$V(K)$ = # vert. of K

$E(K)$ = # edges of K

$F(K)$ = " faces " "

Ex: i) $\chi(pt) = 1$

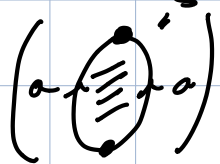
ii) $\chi\Big($  $\Big) = 12 - 12 + 1 = 1$

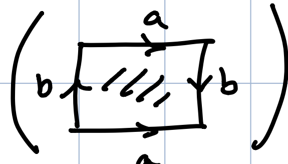
iii) $\chi\Big($ 

$\Big) = 1 - 2 + 1 = 0$

$\chi\Big($  $\Big)$

S^2

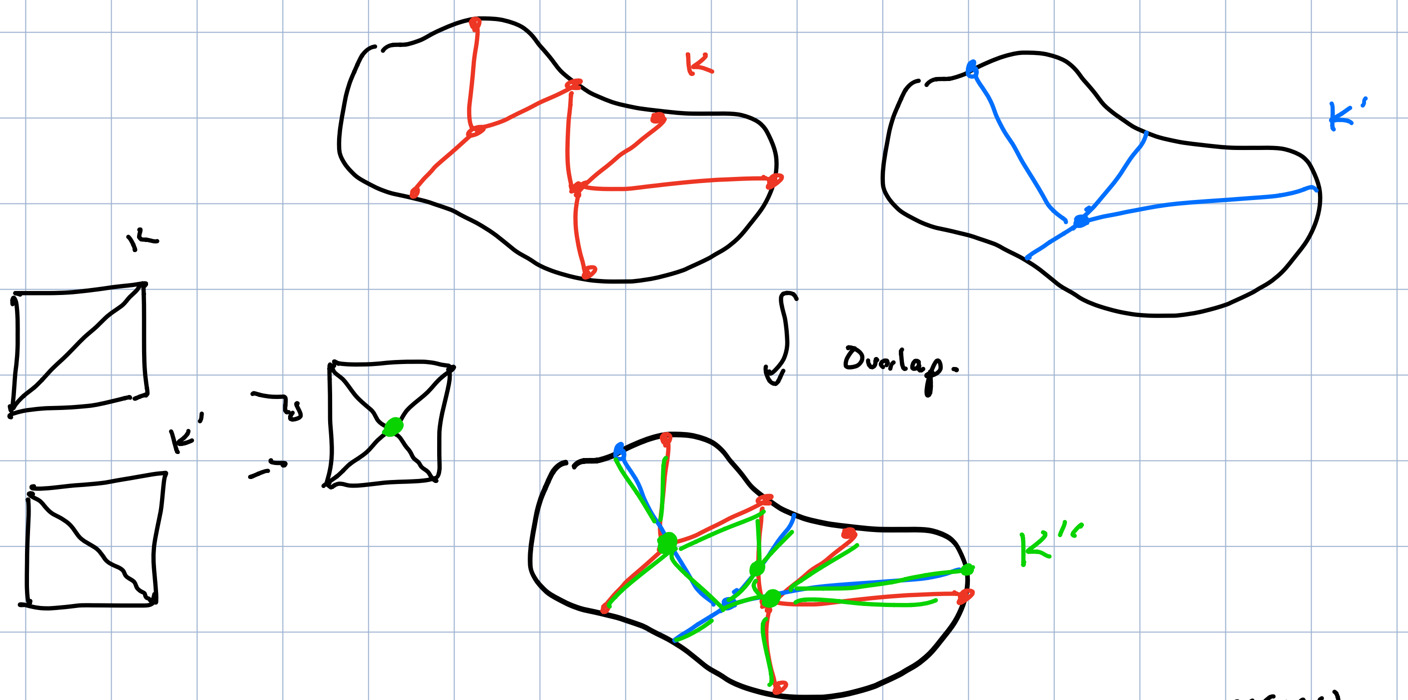
iv) $\chi\Big($  $\Big) = 2 - 1 + 1 = 2$

v) $\chi\Big($  $\Big) = 1 - 2 + 1 = 0$

Defn: $\chi(X)$, where X = surf, equal to $\chi(K)$ where K is any poly. structure homeo to X.

Thm: $\chi(X)$ is ind. of the choice of K.

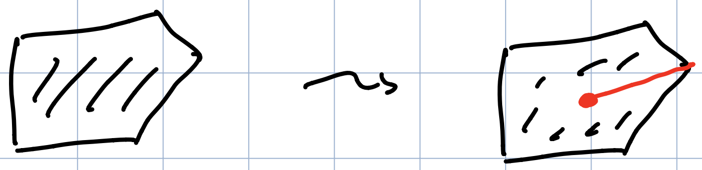
Proof: Let K, K' be any two poly cpx homeo to X. Overlap K w/ K' to obtain a new poly cpx K''



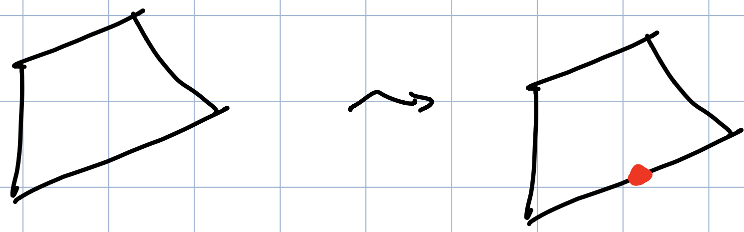
So to prove the claim it suff to show $\chi(K'') \overset{\chi(K')}{=} \chi(K)$.

We obtain K'' from K via applying the following op. in a seq.

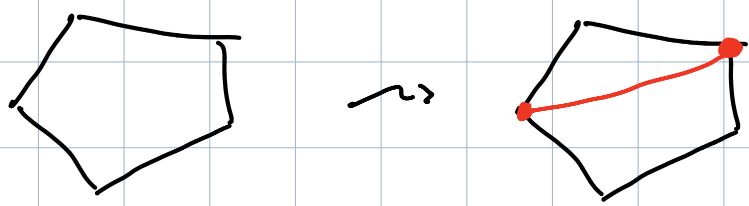
1)



2)



3)



So we just need that these steps

1) $\chi(\text{LHS }(1)) = 1$, $\chi(\text{RHS }(1)) = 1$

$\hookrightarrow$ new vert cancels new edge.

2) same logic applies

3) dividing face cancels the new edge creation

$\Rightarrow$ these steps don't change χ.

$\Rightarrow \chi(K'') = \chi(K)$

$\square$

Thm: S^2 is not homeomorphic to T^2.

Proof: If S^2 is homeo to T^2, then poly cpx for S^2 gives a poly cpx for T^2 w/out changing # V, E, F.

$\Rightarrow 2 = \chi(S^2) = \chi(T^2) = 0$

Contradiction $\Rightarrow$ original assumption must have been wrong. $\Rightarrow S^2 \not\cong T^2$.

$\square$

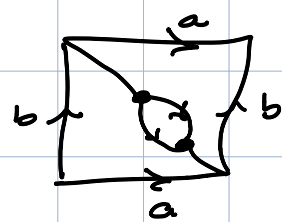
Fact: $\chi(X_1 \# X_2) = \chi(X_1) + \chi(X_2) - 2$

Ex: i) $S^2 \# S^2 = S^2$

$$\chi(S^2 \# S^2) = \chi(S^2) + \chi(S^2) - 2$$
$$= 2 + 2 - 2$$
$$= 2.$$

✓

Proof: Pick poly cpx for X_1, X_2 that have



in the triangulation

So to "remove" our disk from X_1 (resp X_2), we remove the face of this bigon.

X_2

↓ remove disk

$X_2 - \text{disk}$

We bigon along the boundaries to glue up X_1 w/ X_2 to get $X_1 \# X_2$.

$\chi(X_1 \# X_2)$

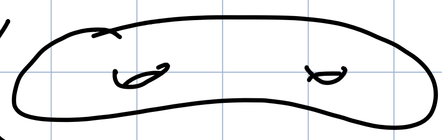
$$V(X_1 \# X_2) = V(X_1) + V(X_2) - 2$$

$$E(X_1 \# X_2) = E(X_1) + E(X_2) - 2$$

$$F(X_1 \# X_2) = F(X_1) - 1 + F(X_2) - 1$$

$$= F(X_1) + F(X_2) - 2$$

$$\Rightarrow \chi(X_1 \# X_2) = \chi(X_1) + \chi(X_2) - 2 \qquad \square$$

Ex: $\chi\Big($  $\Big) = \chi(T^2) + \chi(T^2) - 2$

$= 0 + 0 - 2$

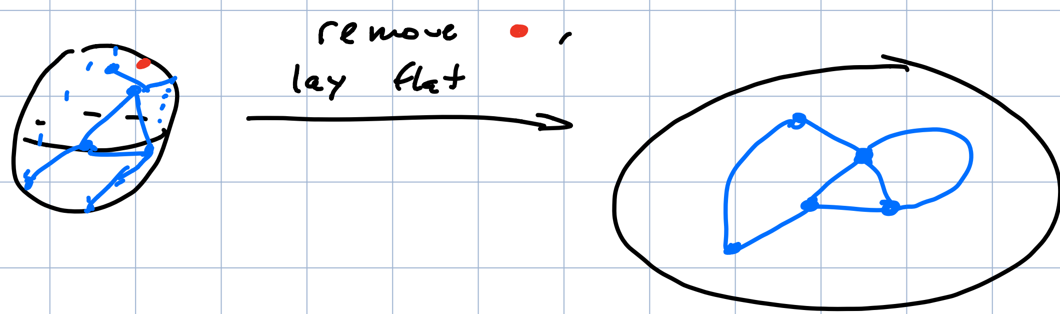
$= -2$

$\chi\Big($  $\Big) = \chi\Big($  $\Big) + \chi(T^2) - 2$

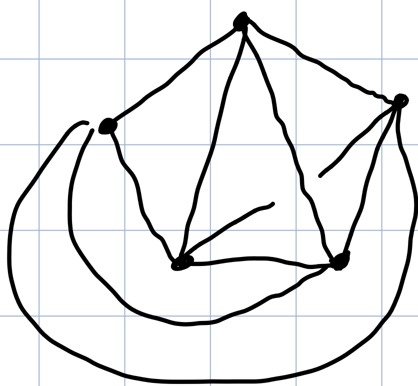
$= -2 + 0 - 2$

$= -4$

Defn: A graph is planar if it can be realized as the edges of a poly. cpx on a sphere.



Defn: K_5 = graph w/ 5 vertices and each pair of vert is connected via unique edge.



$= K_5$

$V(K_5) = 5$

$E(K_5) = 10$

Claim: K_5 is not planar.

Proof: Spse by way of contra. that K_5 is planar

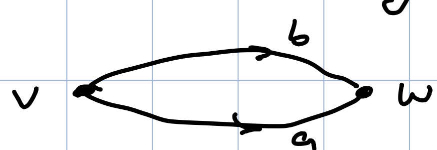
=> realize K_5 as edges on sphere.

$$2 = \chi(S^2) = V(K_5) - E(K_5) + F$$
$$= 5 - 10 + F$$

$F = 7$

Notice that each faces has at least 3 edges.

↳ if not, then we see bigon



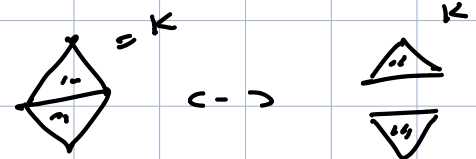
=> v and w are conn. via more than one edge

which doesn't occur

=>

$\Rightarrow \quad 21 = 3F \leq 2E = 20$

K = cpx on S^2, $\tilde{K}$ denote the cpx before we glued to obtain K.



$$E(\tilde{K}) = 2E(K)$$
$$F(\tilde{K}) = F(K)$$
$$3F(\tilde{K}) \leq E(\tilde{K})$$

$\Rightarrow \quad 21 = 7 \cdot 3 \leq E(\tilde{K}) = 2E(K) = 20$

$\Rightarrow$ contradiction!

$\Rightarrow$ original ass. wrong. i.e K_5 is not planar. $\square$

2.4.25: $\chi(g \text{ holes}) = 2 - 2g$

2.4.26

construct planar dgm for a intertube w/ g-holes.

$$\chi(X \# T^2) = \chi(X) + \overset{=0}{\chi(T^2)} - 2$$

$$2 - 2g.$$

Coloring Thm for maps.

$$\# \text{ colors to any map on } S^2 = 4$$

$$\# \text{ colors} \leq \frac{7 + \sqrt{49 - 24\chi(X)}}{2}$$