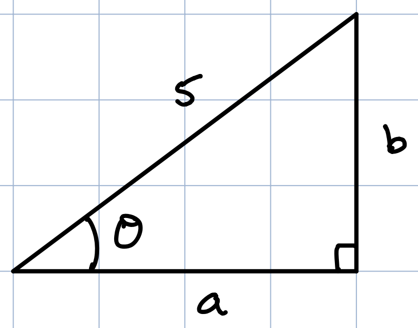
Lecture #20

Warm-up: Consider the triangle



If $\tan(\theta) = \frac{1}{2}$, what are the values of a and b?

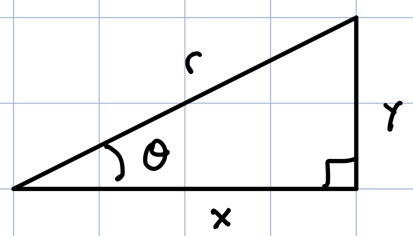
↳ $\tan(\theta) = b/a \Rightarrow \frac{1}{2} = \frac{b}{a} \Rightarrow a = 2b.$

$25 = a^2 + b^2 = (2b)^2 + b^2 = 5b^2$

$\Rightarrow b = \sqrt{5}$

$\Rightarrow a = 2\sqrt{5}$

Recall: Given a right triangle



$\Rightarrow$

$\sin(\theta) = y/r$

$\cos(\theta) = x/r$

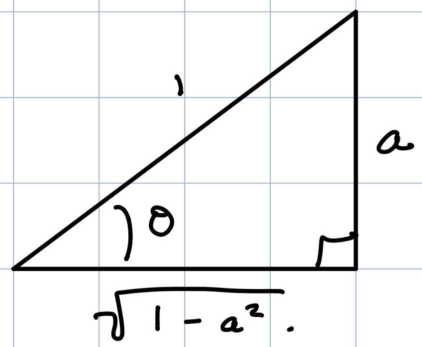
$\tan(\theta) = y/x$

$r^2 = y^2 + x^2$

↳ SOHCAHTOA

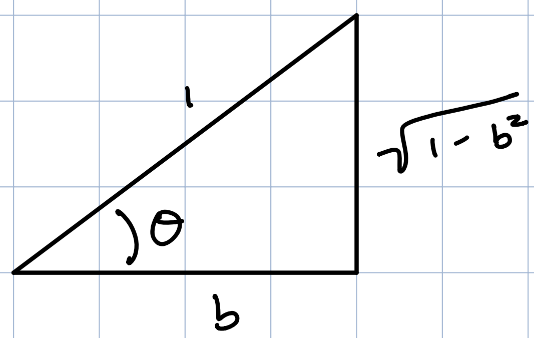
Section 6.4: Inverse Trig via Right Δs.

Rmk:



$\sin^{-1}(a) = \theta$

$\Rightarrow \sin(\theta) = a$



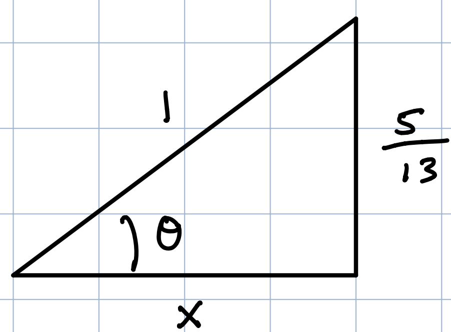
$\cos^{-1}(b) = \theta$

$\Rightarrow \cos(\theta) = b$

Ex: $\cos(\sin^{-1}(5/13))$

① Set $\theta = \sin^{-1}(5/13)$

② Form a right Δ w/ angle θ, hypo 1, height $\frac{5}{13}$



$x^2 + \left(\frac{5}{13}\right)^2 = 1^2$

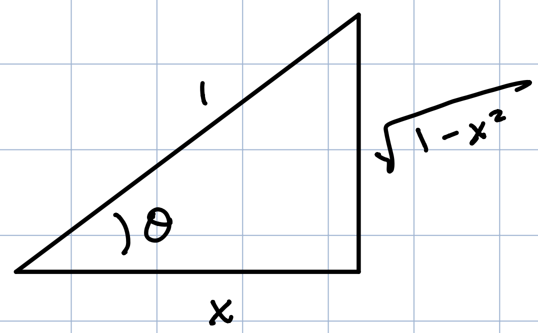
③ Compute $\cos(\theta) = \cos(\sin^{-1}(5/13))$ using the right Δ.

$$\cos(\theta) = x = \sqrt{1 - (5/13)^2}$$

Pythag. Thm

Ex: Write $\tan(\cos^{-1}(x))$ w/ out using trig fcns.

$\theta = \cos^{-1}(x)$.

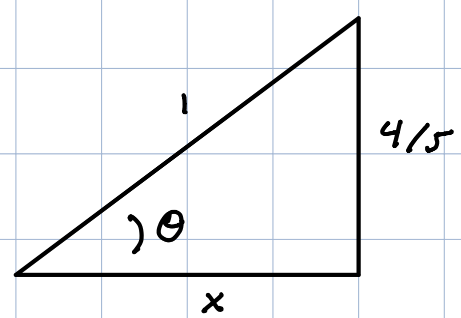


$\Rightarrow \tan(\cos^{-1}(x)) = \tan(\theta) = \sqrt{1-x^2}/x$.

Ex: Compute $\cos(\sin^{-1}(4/5)) \overset{?}{=} \cos(\theta) = x = \frac{3}{5}$

$\theta = \sin^{-1}(4/5)$

$\sin(\theta) = 4/5$



By Pathag. Thm, $1^2 = \left(\frac{4}{5}\right)^2 + x^2$.

$\Rightarrow x = \sqrt{1-(4/5)^2} = \sqrt{\frac{25}{25} - \frac{16}{25}} = \sqrt{9/25} = 3/5$.

Ch 7: Algebra of Trign. Fcns.

Defn: An equation is a statement of equality

↳ $x^2 - 2x + 1 = 0$, $x = 1$, $\frac{6}{2} = 3$.

An identity is a statement of equality that holds for all values of the variables involved.

↳ Ex: $\sin^2(t) + \cos^2(t) = 1$.

Ex: $|x|^2 = x^2$.

Non-ex: $x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

$\Rightarrow x = 1$.

Fact:

1) (Pythag.) $\sin^2 t + \cos^2 t = 1$

2) (even-odd) $\sin(-t) = -\sin(t)$ (sin is odd)

$\cos(-t) = \cos(t)$ (cos is even).

3) (cofunction) $\sin(x + \pi/2) = \cos(x)$.

Ex: We can use these id's to simplify expressions.

i) $$\frac{\sec(t) - \cos(t)}{\sin(t)} = \frac{\frac{1}{\cos(t)} - \cos(t)}{\sin(t)} \cdot \frac{\cos(t)}{\cos(t)}$$

$$= \frac{1 - \cos^2(t)}{\sin(t)\cos(t)}.$$

i) above

$$= \frac{\sin^2(t)}{\sin(t)\cos(t)}$$

$$= \sin(t)/\cos(t)$$

$$= \tan(t).$$

Ex:
$$\frac{1+\sin(t)}{\cos(t)} + \frac{\cos(t)}{1+\sin(t)}$$

① $$= \frac{1+\sin(t)}{\cos(t)} \cdot \frac{1+\sin(t)}{1+\sin(t)} + \frac{\cos(t)}{1+\sin(t)} \cdot \frac{\cos(t)}{\cos(t)}$$

② $$= \frac{1 + 2\sin(t) + \sin^2(t) + \cos^2(t)}{\cos(t)(1+\sin(t))}$$

$\sin^2(t) + \cos^2(t) = 1$

③ $$= \frac{2 + 2\sin(t)}{\cos(t)(1+\sin(t))}$$

④ $$= \frac{2(1+\sin(t))}{\cos(t)(1+\sin(t))}$$

$$= 2/\cos(t)$$

$$= 2\sec(t).$$

Ex: We can use these id's to produce new identities

i) $1 + \tan^2(t) = \sec^2(t)$.

$$\Leftrightarrow \cos^2(t) + \sin^2(t) = 1$$

$$\Rightarrow \frac{\cos^2(t)}{\cos^2(t)} + \frac{\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)}$$

$$\Rightarrow 1 + \tan^2(t) = \sec^2(t).$$

$1 = \sec^2(t) - \tan^2(t).$

ii) $\sec^4(t) - \tan^4(t) = \sec^2(t) + \tan^2(t).$

$$= (\sec^2(t) + \tan^2(t))(\sec^2(t) - \tan^2(t))$$

$$= \sec^2(t) + \tan^2(t).$$

Tips:

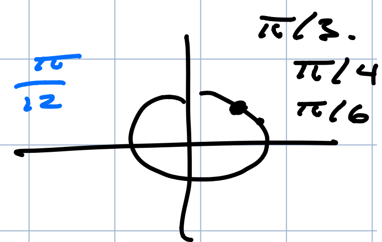
1) If proving new identity, then start w/ more complicated side of eqn.
2) Write items in terms of cos, sin when stuck.
3) Apply previous methods to simplify
   ↳ combine fractions, factor, etc.

Section 7.2.

Fact:

1) $\sin(s+t) = \sin(s)\cos(t) + \cos(s)\sin(t)$.
2) $\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$.

$\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$



Ex: Compute $\cos(\pi/12)$

$$
\begin{aligned}
\cos(\pi/12) &= \cos(\pi/4 - \pi/6) \\
&= \cos(\pi/4)\cos(-\pi/6) - \sin(\pi/4)\sin(-\pi/6). \\
&= (\sqrt{2}/2)(\sqrt{3}/2) - (\sqrt{2}/2)(-1/2) \\
&= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}
$$

$$\cos(s+t) = \cos(s)\cos(-t) - \sin(s)\sin(-t)$$

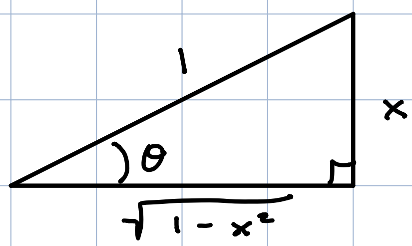
Ex: Write $\cos(\sin^{-1}(x) + \tan^{-1}(y))$ in terms of x, y and no trig fcns.

First, by sum formula for cos.

$$\cos(\underbrace{\sin^{-1}(x)}_{s} + \underbrace{\tan^{-1}(y)}_{t})$$

$$= \overset{①}{\cos}(\sin^{-1}(x))\ \overset{②}{\cos}(\tan^{-1}(y)) - \overset{③}{\sin}(\sin^{-1}(x))\ \overset{④}{\sin}(\tan^{-1}(y)).$$

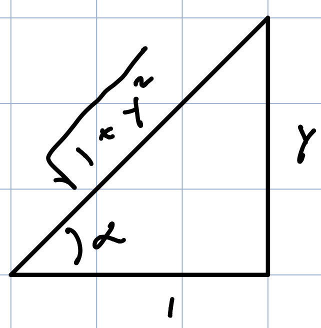
$\theta = \sin^{-1}(x)$



① $= \sqrt{1-x^2}$

③ $= x$

$\alpha = \tan^{-1}(y) \implies \tan(\alpha) = y$



② $= \dfrac{1}{\sqrt{1+y^2}}$.

④ $= \sin(\alpha)$

$= y / \sqrt{1+y^2}$

Section 7.3:

Fact: (Double Angle Formula)

i) $\sin(2x) = \sin(x + x)$ ↓ sum angle form.

$= \sin(x)\cos(x) + \sin(x)\cos(x)$

$= 2\sin(x)\cos(x).$

ii) $\cos(2x) = 2\cos^2(x) - 1.$

Ex: Simplify

$$\frac{\sin(2x)}{\cos^2(x)} = \frac{2\sin(x)\cos(x)}{\cos^2(x)} = 2\tan(x).$$