Lecture # 15

Ch. 4: Exponential functions and logarithmic fcns

Defn: The exponential fcn w/ base a $(a \neq 0, a > 0)$ is

$$f(x) = a^x$$

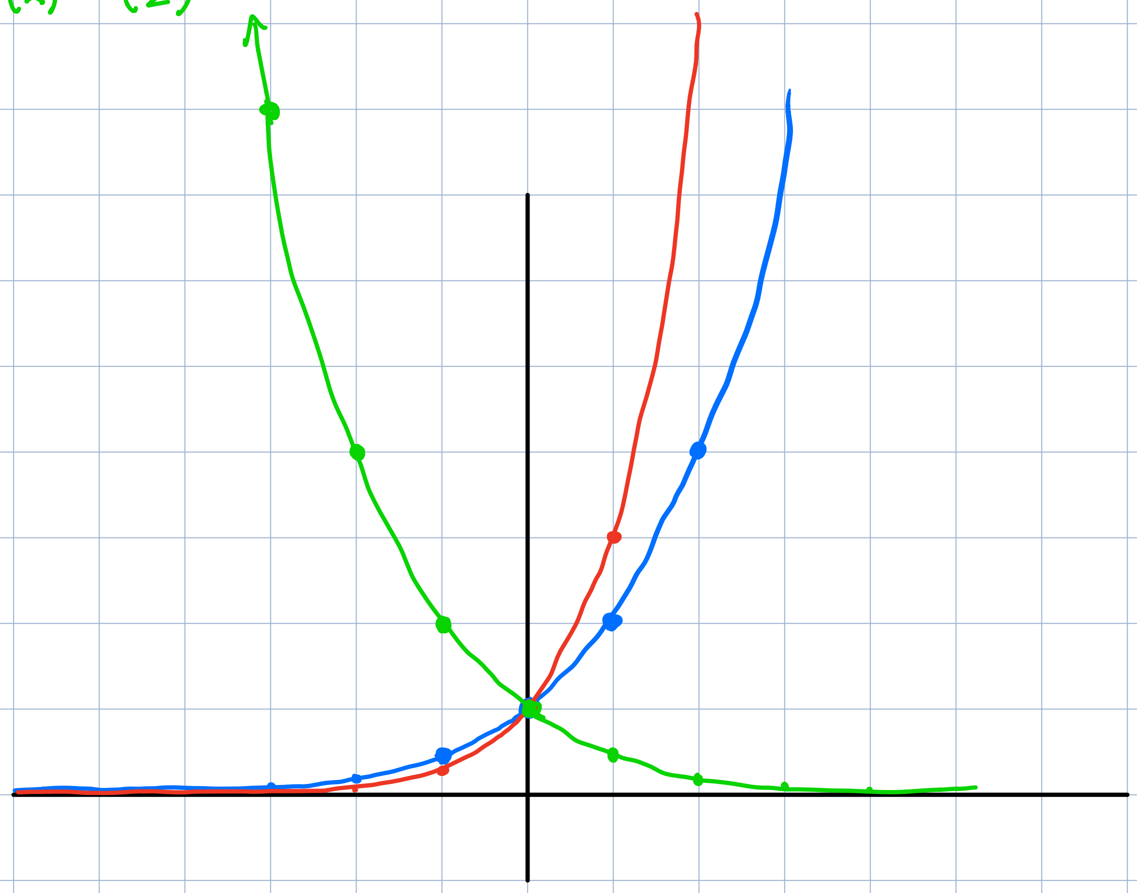
Ex: $a = 2$, $a = 3$, $a = \frac{1}{2}$

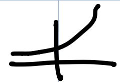
$f(x) = 2^x$

$f(x) = 3^x$

$f(x) = \left(\frac{1}{2}\right)^x$

$$2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$$



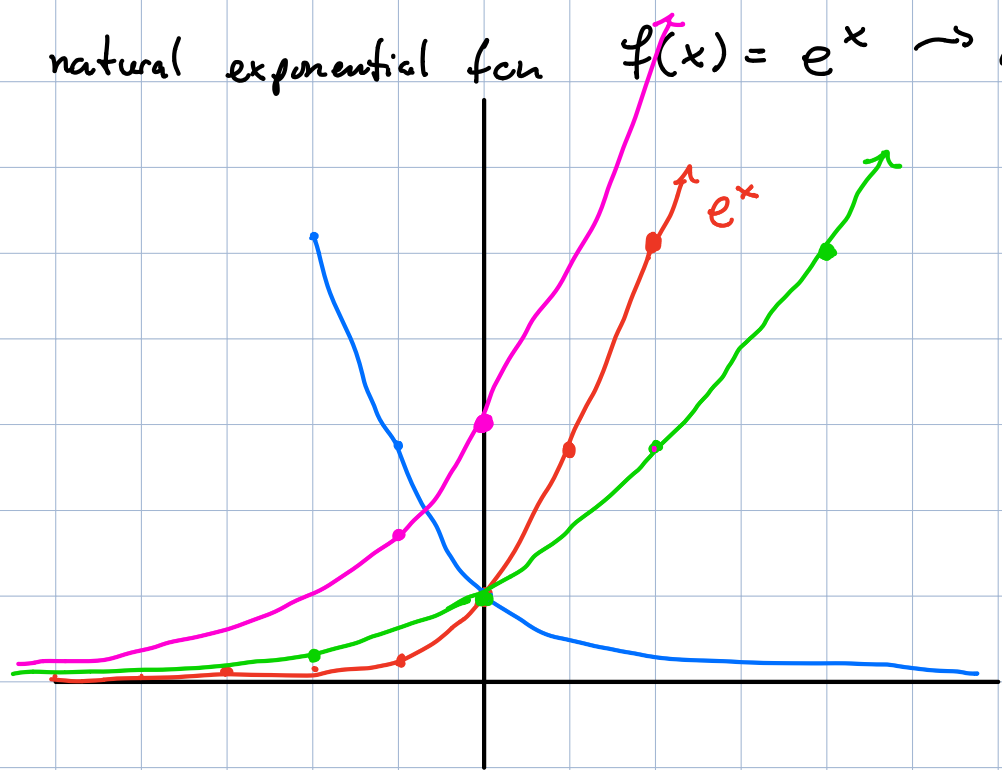
Remark:
- $a > 1 \Rightarrow$ 
- $a < 1 \Rightarrow$ 
- In gen, a^x has a horizontal asy. along the x-axis
- $\text{Range}(a^x) = (0, +\infty)$
- $\text{dom}(a^x) = \mathbb{R}$

Section 4.2: Expontial fcn

Defn: $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828\ldots\ldots$

↳ irrational #. (like π, $\sqrt{2}$).

Defn: The natural exponential fcn $f(x) = e^x$ ⟶ graph w/ calc.



$f(x) = e^{-x}$

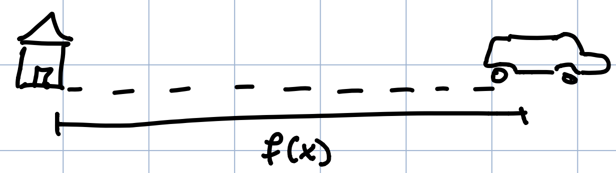
$f(x) = e^{x/2}$

$f(x) = 3e^{x/2}$

$x \to c \cdot x$

$\Rightarrow$ hor. stretch by $\frac{1}{c}$

Remark: Spse that $f(x)$ = dist a car travels at time x



Spse that speed of the car at time x is $f(x)$.
Then $f(x) = e^x$.

Section 4.3: Logarithmic Functions.

Rmk: By the hor. line test, $f(x) = a^x$ is 1-to-1
$\Rightarrow$ a^x has an inverse functions

Defn: The logarithmic fcn w/ base a $(a>0,\ a\neq 1)$ is

$$\log_a : (0,\infty) \longrightarrow \mathbb{R}$$

given by $\log_a(x) = y$ st $a^y = x$.

Ex:

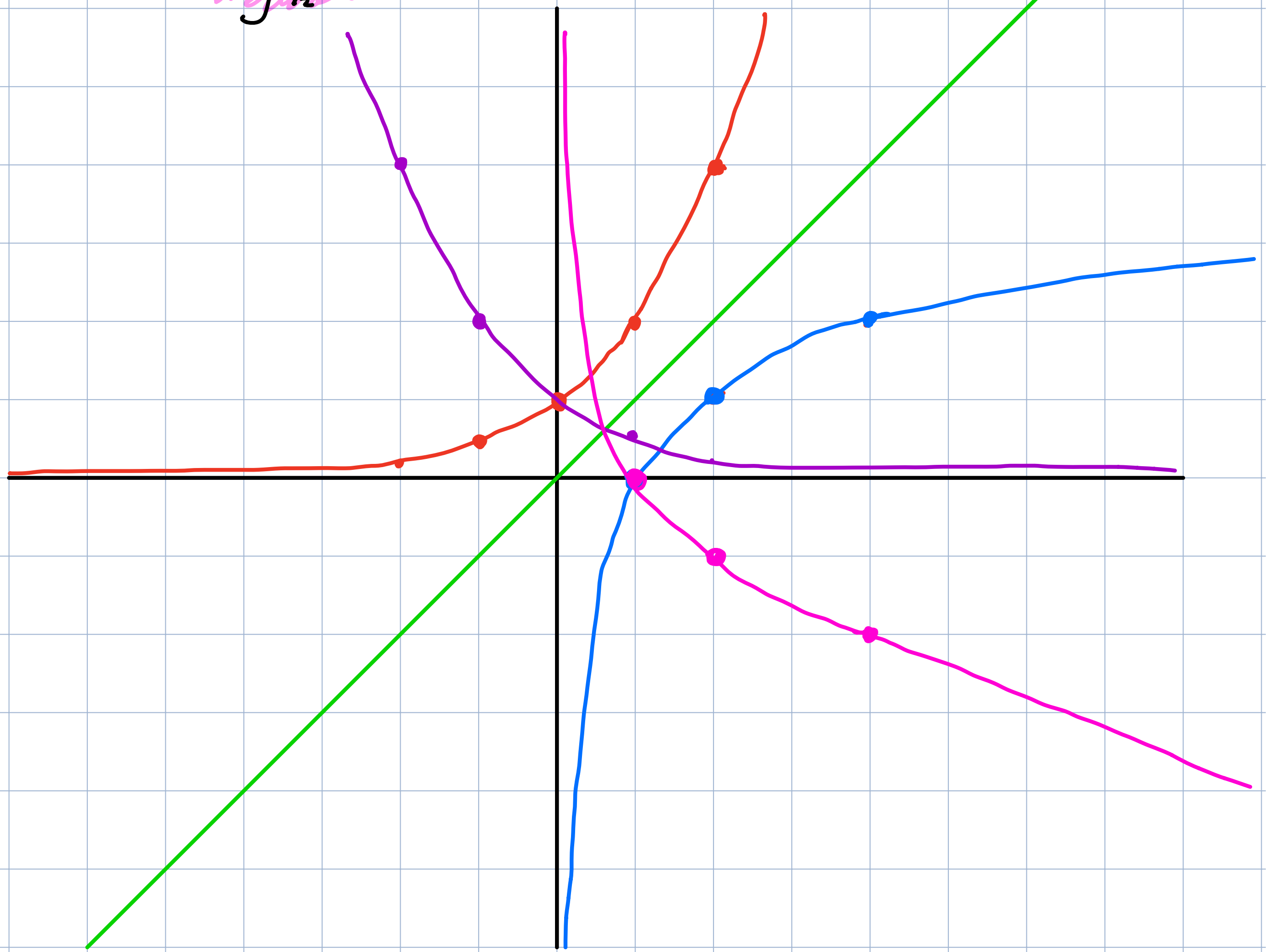
- $\log_2(2) = y$ st $2^y = 2$ $\Rightarrow$ $y=1$
- $\log_2(8) = y$ st $2^y = 8$ $\Rightarrow$ $y=3$
- $\log_{42}(1) = y$ st $42^y = 1$ $\Rightarrow$ $y=0$
- $\log_e(e^2) = y$ st $e^y = e^2$ $\Rightarrow$ $y=2$

Ex: $f(x) = 2^x$

$f(x) = \left(\frac{1}{2}\right)^x$

$f(x) = \log_2(x)$

$f(x) = \log_{1/2}(x)$



Remark: $x \to 0, \quad y \to -\infty$ } $a > 1$

$x \to \infty, \quad y \to +\infty$

$x \to 0, \quad y \to +\infty$ } $a < 1$

$x \to +\infty, \quad y \to -\infty$

Rmk: log has ver. asy along y-axis.

Fact: 1) $\log_a(1) = 0$ $= y$ st $a^y = 1 \Rightarrow y = 0$; $a^0 = 1$

2) $\log_a(a) = 1$ $= y$ st $a^y = a \Rightarrow y = 1$; $a^1 = a$

3) $\log_a(a^x) = x$ $= y$ st $a^y = a^x \Rightarrow y = x$

4) $a^{\log_a(x)} = x$ $=$ sim.

↳ (3), (4) a^x, $\log_a(x)$ are inverses.

Defn: The natural logarithm is $\log_e(x) = \ln(x)$.

Ex: What is the domain of

$$f(x) = \frac{\ln(4-x^2)}{x-3}$$

1) Can't divide by zero $\Rightarrow x \neq 3$.

2) ln(neg. #s) is not defn.

$\Rightarrow 4 - x^2 > 0$

$\Rightarrow 4 > x^2$

$\Rightarrow -2 < x < 2$

$\Rightarrow \text{Dom}(f) = \{x \mid -2 < x < 2\}$.

Section 4.4: Laws of Logarithms

Fact: 1) $\log_a(A \cdot B) = \log_a(A) + \log_a(B)$

↳ raise a to the power of LHS and RHS and comp

$$\left.\begin{aligned} a^{\log_a(A\cdot B)} &= A \cdot B \\ a^{\log_a(A) + \log_a(B)} &= a^{\log_a(A)} \cdot a^{\log_a(B)} = A \cdot B \end{aligned}\right\} \Rightarrow \text{agree.}$$

2) $\log_a(A/B) = \log_a(A) - \log_a(B)$

3) $\log_a(A^C) = C \cdot \log_a(A)$

Ex: i) $\ln\left(\frac{a^2 b^4}{\sqrt[3]{c}}\right) = \ln(a^2 b^4) - \ln(\sqrt[3]{c})$ ②

$= \ln(a^2) + \ln(b^4) - \ln(c^{1/3})$ ①

$= 2\ln(a) + 4\ln(b) - \frac{1}{3}\ln(c).$ ③

Ex: $3\ln(s) - 5\ln(s-2) + \ln(t^2-5)$

$= \ln(s^3) - \ln((s-2)^5) + \ln(t^2-5)$ ③

$= \ln\left(\frac{s^3}{(s-2)^5}\right) + \ln(t^2-5)$ ②

$= \ln\left(\frac{s^3(t^2-5)}{(s-2)^5}\right)$ ①

Fact: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

↳ book for proof of this fact.

Section 4.5: Exponential and Log eqn.

Ex: i) $5^x = 600$, solve for x

$\Rightarrow \log_5(5^x) = \log_5(600)$

$\Rightarrow x = \log_5(600)$

ii) $3^x = 27\sqrt{3}$

$\Rightarrow \log_3(3^x) = \log_3(27\sqrt{3})$

$\Rightarrow x = \log_3(27\sqrt{3})$

$= \log_3(27) + \log_3(\sqrt{3})$

$= \log_3(3^3) + \log_3(3^{1/2})$

$= 3 + 1/2$

$= 3.5$

iii) $2^{x+2} = 2^{2x+3}$

$\Rightarrow \log_2(2^{x+2}) = \log_2(2^{2x+3})$

$\Rightarrow x+2 = 2x+3$

$\Rightarrow 0 = x+1$

$\Rightarrow x = -1.$

Ex: i) $\log_2(25-x) = 3$

$\Rightarrow 2^{\log_2(25-x)} = 2^3$

$\Rightarrow 25 - x = 8$

$\Rightarrow x = 17$

(We check that $\log_2(25-17)$ is defn $\Rightarrow x = 17$ <u>is</u> in fact a solution).

ii) $\log(x-2) + \log(x-1) = \log(x-3)$

$\Rightarrow e^{\log(x-2)+\log(x-1)} = e^{\log(x-3)}$

$\Rightarrow (x-2)(x-1) = x-3$

$\Rightarrow x^2 - 3x + 2 = x - 3$

$\Rightarrow x^2 - 4x + 5 = 0$

discr $= 16 - 4(5) = 16 - 20 = -4$

$\Rightarrow$ no real solutions