Lecture # 14

Warm-up: 1) Find a poly. w/ real coeffs that has zeros at $x = 1$ and $x = -i$ and satisfies $P(0) = 1$ y-int = (0, 1).

↳ Real coeff ⟹ cpx roots occur as conj pairs.

ie $a+ib$ = root ⟶ $a-ib$ = root = $\overline{a+ib}$

$P(x) = a\cdot(x-1)(x+i)(x-i)$ $\qquad i^2=-1$

$= a(x-1)(x^2+xi-xi-i^2)$

$= a(x-1)(x^2+1)$

$= a(x^3-x^2+x-1)$

$1 = P(0) = a\cdot(-1)$

$\Rightarrow a = -1$

$\Rightarrow P(x) = -(x^3-x^2+x-1)$.

2) Completely factor the polynomial

$$P(x) = x^3 - 2x^2 + 4x - 8$$

↳ Rat'l root test tell us that only poss rat'l zeros are $\pm 1, \pm 2, \pm 4, \pm 8$.

$x = 2$

$P(2) = 8 - 2(4) + 4(2) - 8 = 0$

$\Rightarrow (x-2)$ is a factor

Apply Long div

$$\begin{array}{r} x^2 + 4 \\ x-2 \,\overline{)\, x^3 - 2x^2 + 4x - 8} \\ -(x^3 - 2x^2) \\ \hline 4x - 8 \\ -(4x - 8) \\ \hline 0 \end{array}$$

$\Rightarrow \; P(x) = (x-2)(x^2+4)$

$x^2 + 4 = 0 \;\Rightarrow\; x^2 = -4 \;\Rightarrow\; x = \pm\sqrt{-4} = \pm 2i$

$\Rightarrow \; P(x) = (x-2)(x-2i)(x+2i)$.

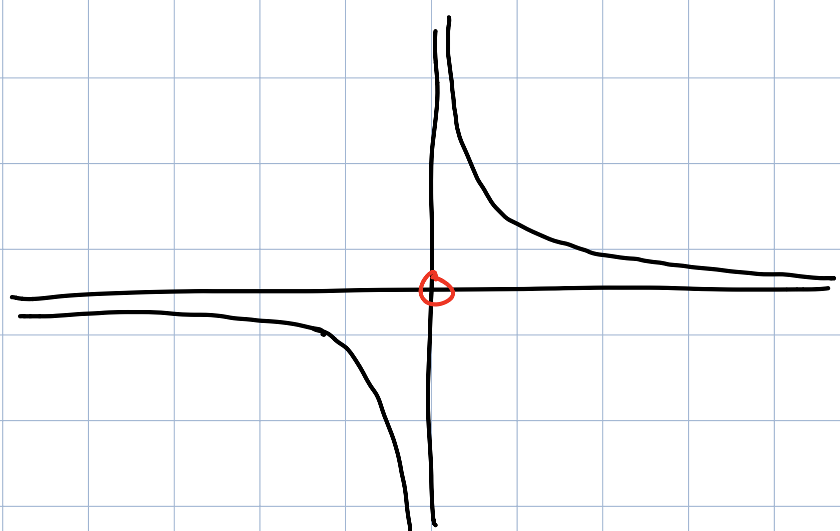
Defn: A rat'l fcn is

$$f(x) = \frac{P(x)}{Q(x)}$$

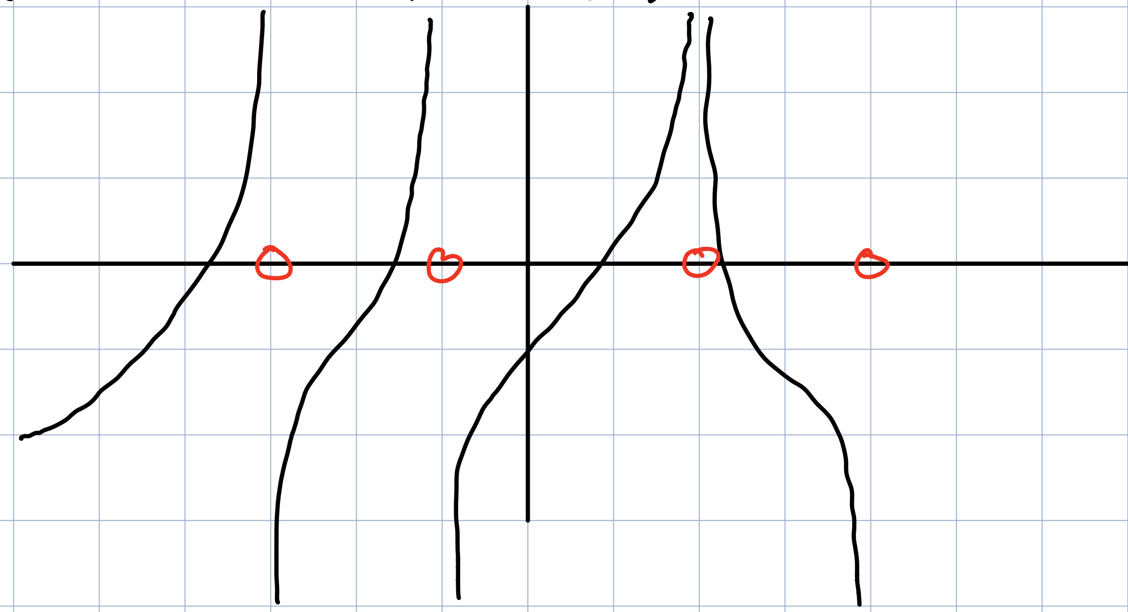
where P, Q poly. w/ no common factors

↳ ie, $f(x) = \dfrac{(x^2-1)(x+1)}{(x^2-3)(x+1)} \neq$ rat'l fcn

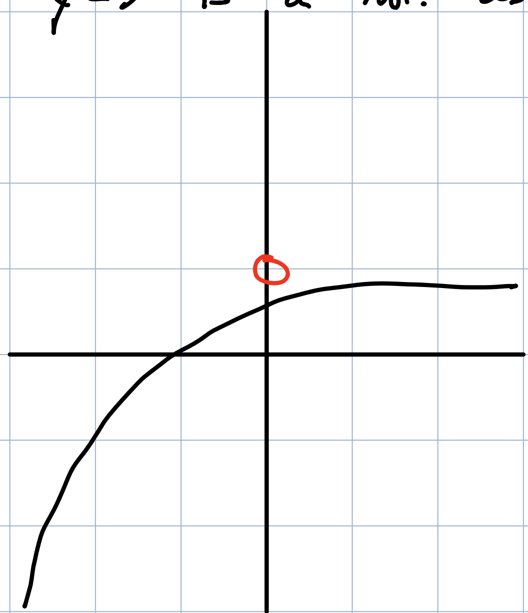
Ex: $f(x) = 1/x$

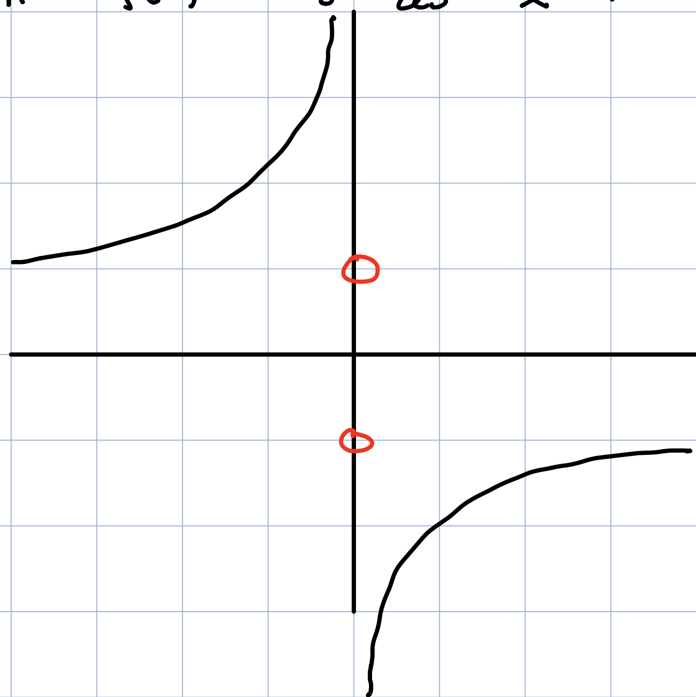


Defn: $x = a$ is a ver. asy. of $f(x)$ if $y \to \pm\infty$ as $x \to a$ (either from left/right)



$y = b$ is a hor. asy if $f(x) \to b$ as $x \to \pm\infty$





Rmk: Ver. asy for rat'l fcn occurs when denom. is equal to zero.

$$\frac{x^{77}}{x^{75}\,x^{2}}$$

$$\frac{5\cancel{x^{77}} - \cancel{6x+2}}{4\cancel{x^{77}} + \cancel{2x^{75}-1}}$$

Rmk: Hor asym occur when

1) $\deg(P) = \deg(Q) \Rightarrow y = \text{leading coef}(P)/\text{lead}(Q)$
2) $\deg(P) > \deg(Q) \Rightarrow$ no hor. asy
3) $\deg(P) < \deg(Q) \Rightarrow y = 0$ hor. asym.

Ex: $f(x) = \dfrac{(2x-1)(x+4)}{(x-1)(x+2)} = \dfrac{2x^2 + 8x - x - 4}{x^2 - x + 2x - 2}$

Dom, range, graph, asy.

$\hookrightarrow$ $\text{Dom}(f) = \{x \mid x \neq 1 \text{ and } x \neq -2\}$

zeros of f = meets x-axis at: $x = \frac{1}{2}$, $x = -4$

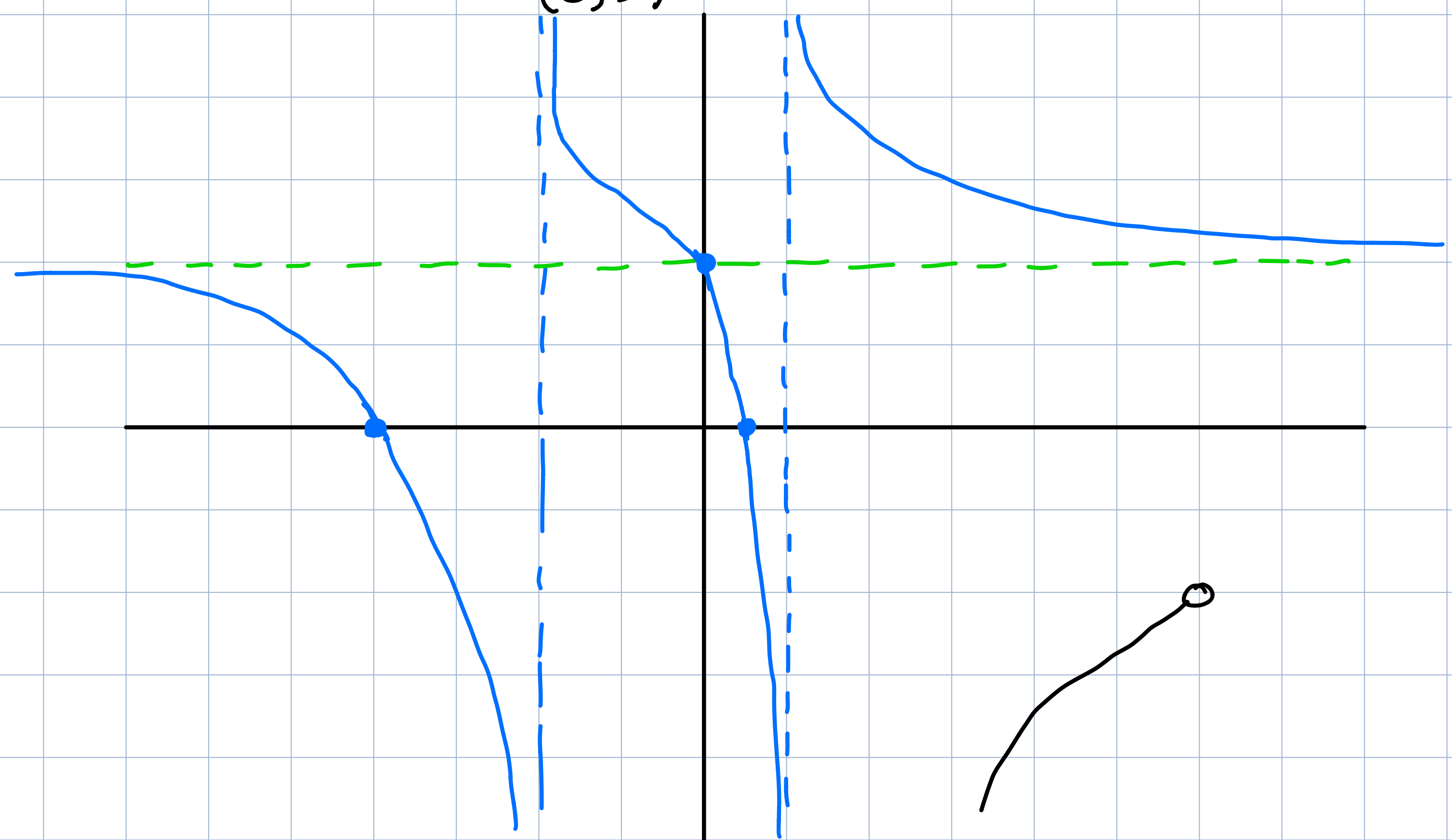
ver asymp = $x = 1$, $x = -2$

hor. asymp = $y = 2/1 = 2$

y-intercept = $(0, f(0))$

$\hookrightarrow$ $f(0) = (-1)(4)/(-1)(2) = 2$

$\Rightarrow (0, 2)$



Range we look at graph (do not include hor. asym in the range)

↳ range $(f) = (-\infty, 2) \cup (2, +\infty)$.

1.001

Ex:

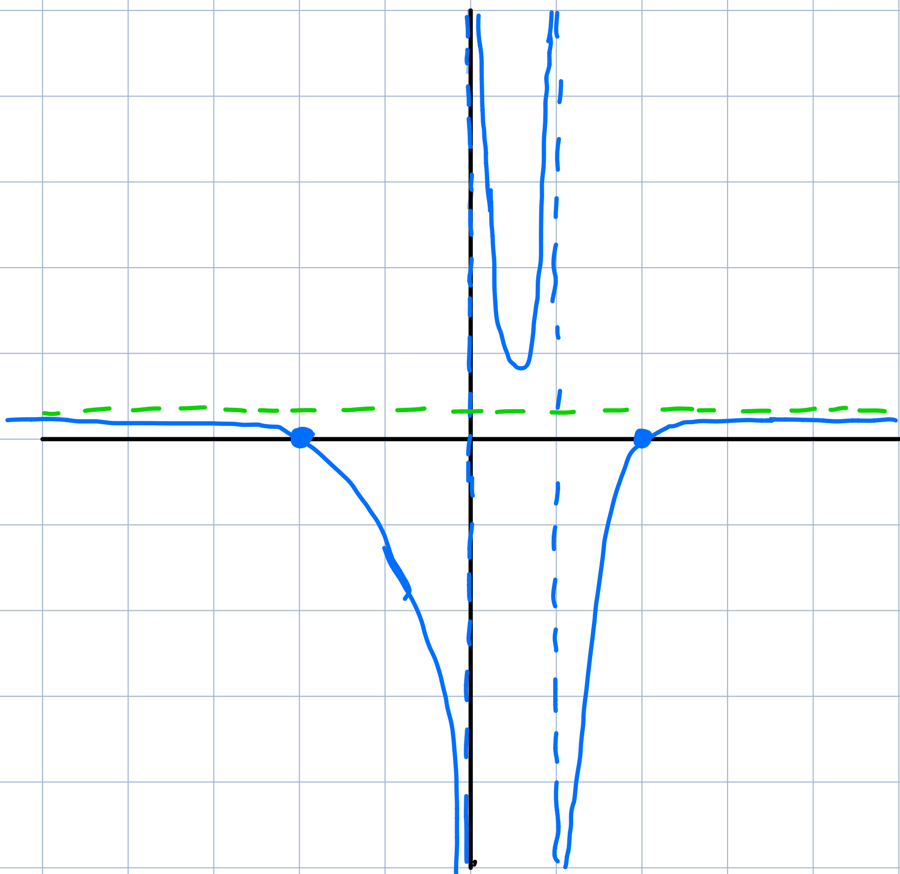
$$f(x) = \frac{x^2-4}{3x^2-3x} = \frac{(x-2)(x+2)}{3x(x-1)}$$

graph.

↳ zeros = zeros of num = $\pm 2 = x$

asy ver = zeros of demon = $x = 0$, $x = 1$

asy hor = $y = 1/3$



Rmk: If

$$f(x) = \frac{P(x)\,(x-c)}{Q(x)\,(x-c)}$$

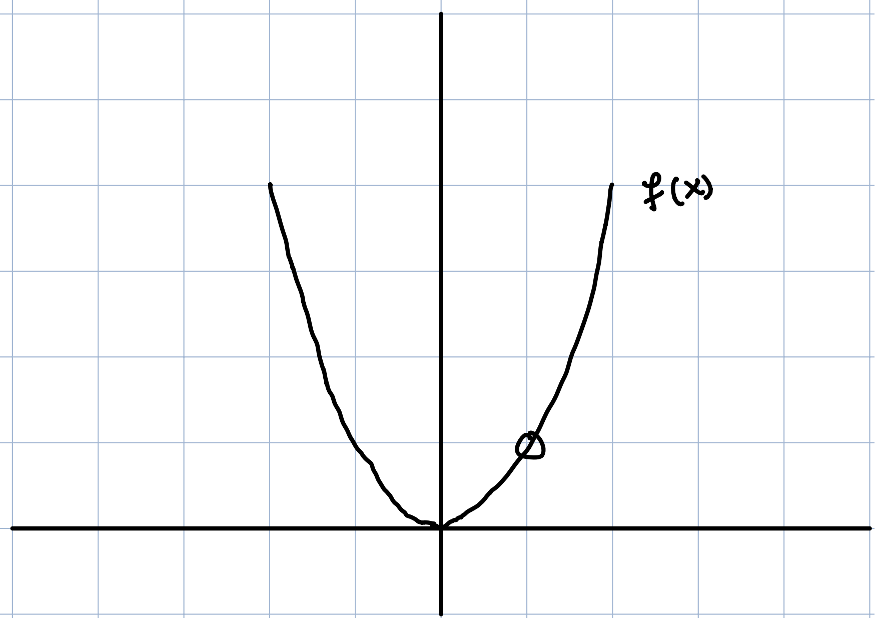
then the graph of f is the graph of P/Q but w/ a hollow dot at $(c, f(c))$

↳ $x = c \Rightarrow f(x)$ is undefined

$x \neq c \Rightarrow (x-c)/(x-c) = 1 \Rightarrow$ doesn't change graph.

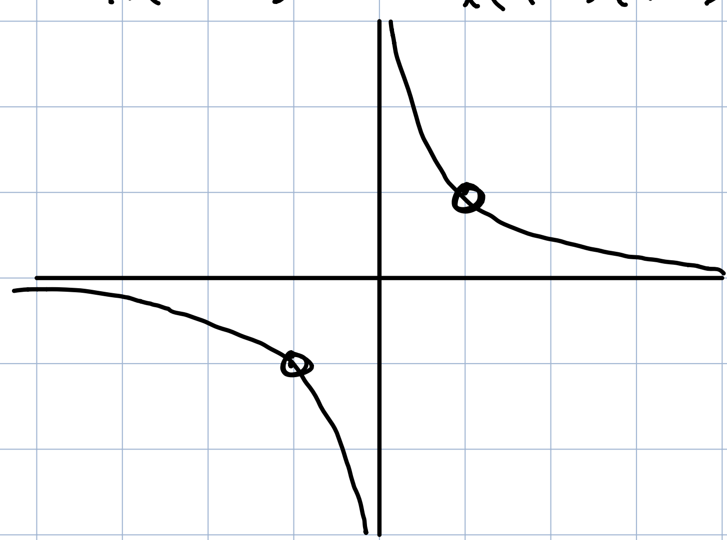
Ex:

$$f(x) = \frac{x^2(x-1)}{(x-1)}$$



Ex:

$$f(x) = \frac{x^2-1}{x(x^2-1)} = \frac{(x-1)(x+1)}{x(x-1)(x+1)}$$



Rmk: $f = P/Q$ where $\deg(P) = \deg(Q)+1$

↳ ie, $\dfrac{x^2-2x+1}{x+3}$

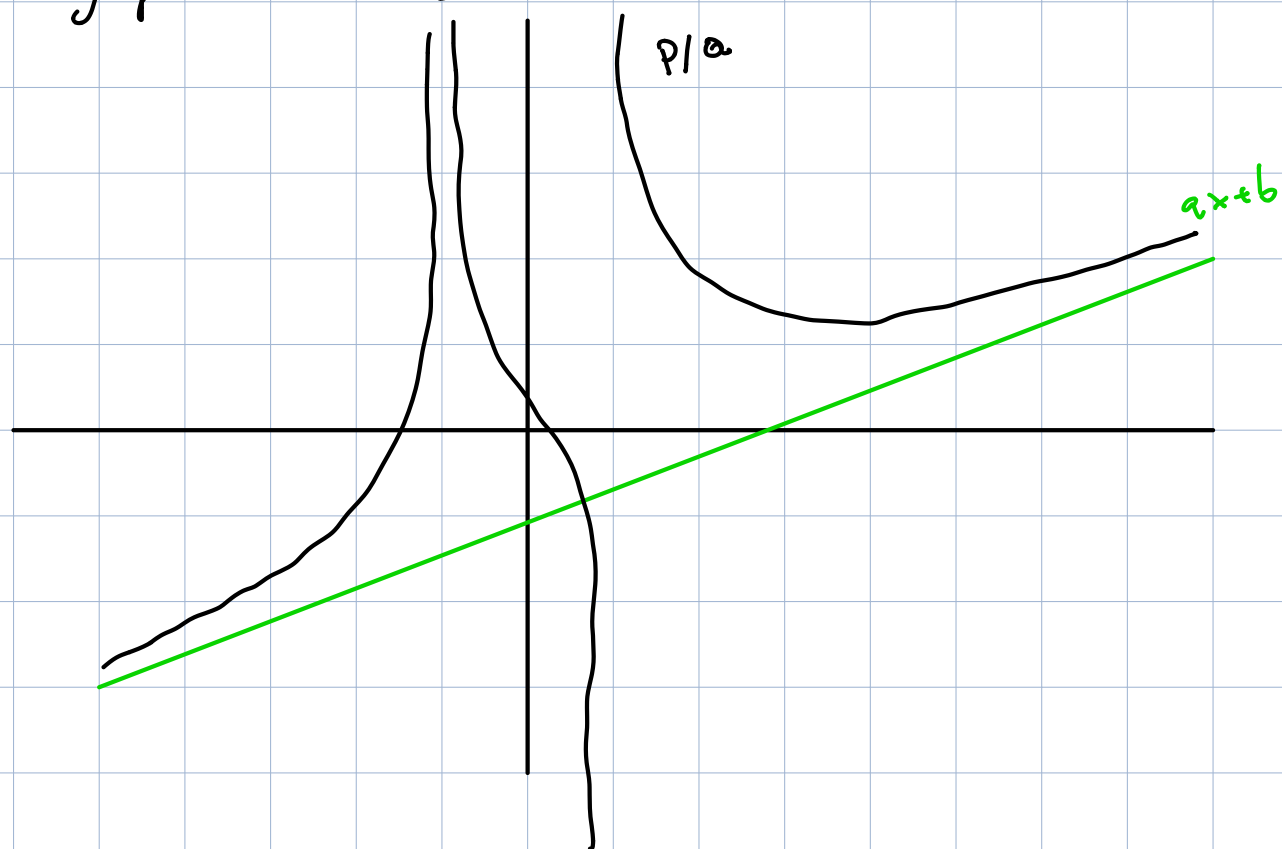
So we can apply long div. to write

$$\frac{P}{Q} = ax+b+\frac{R}{Q}$$

where $\deg(R) < \deg(Q)$

$\Rightarrow$ (hor. asympt table) as $x \to \pm\infty$ $R/Q \longrightarrow 0$

↳ $ax+b$ is a slant asym, ie graph (P/Q) approaches graph of $ax+b$



Ex:

$$f(x) = \frac{x^2-2x+1}{x+3} = \frac{(x-1)^2}{x+3}$$

1) Apply long div.

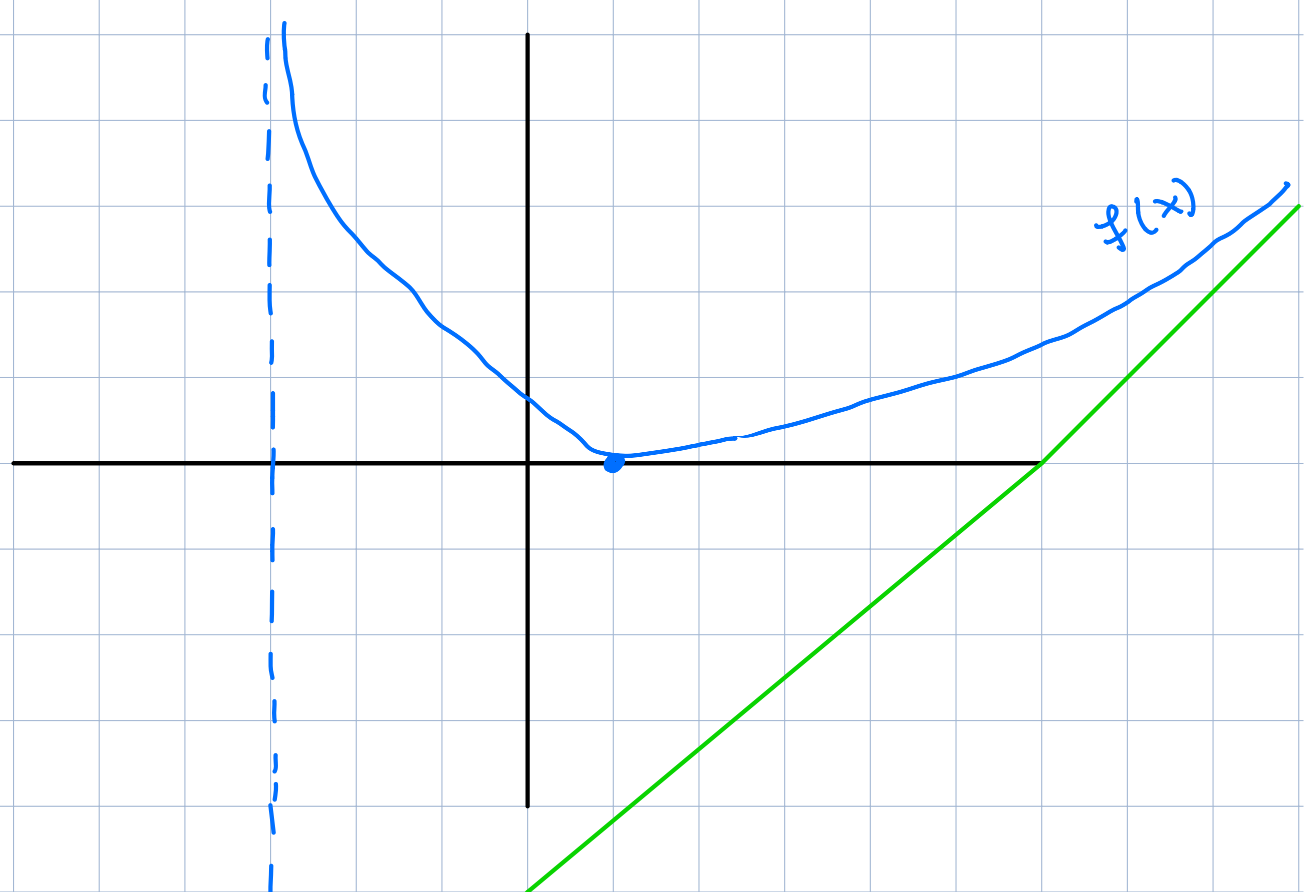
$$\begin{array}{r|l} & x \quad - 5 \\ \hline x+3 & x^2 - 2x + 1 \\ & -(x^2+3x) \\ \hline & \quad -5x + 1 \\ & -(-5x - 15) \\ \hline & \qquad 16 \end{array}$$

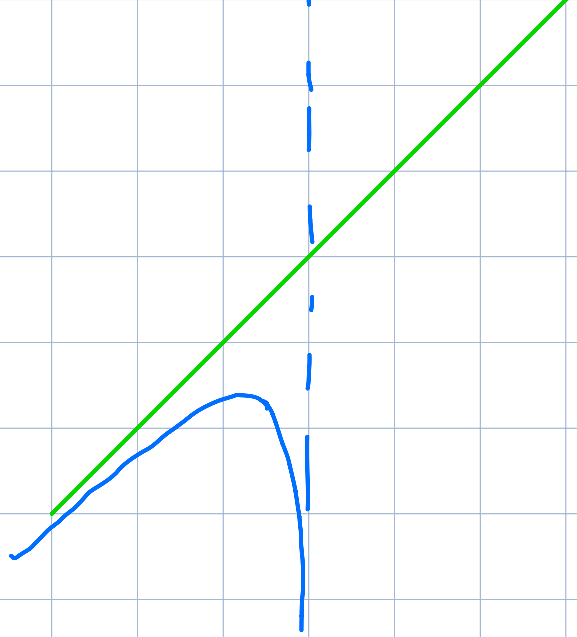
$$\Rightarrow \quad f(x) = x - 5 + \frac{16}{x+3}$$

2) Det ver asymp, x-val where $Q(x) = 0$

$\Rightarrow$ ver. asy $= \quad x = -3$

3) Find zeros of f = x-int. $= x = 1$





Ex:

$$f(x) = \frac{(x-3)(x+4)}{x-2} = \frac{x^2+x-12}{x-2}$$

1) Long Div!

$$\begin{array}{r|l} & x+3 \\ \hline x-2 & x^2+x-12 \\ & -(x^2-2x) \\ \hline & 3x-12 \\ & -(3x-6) \\ \hline & -6 \end{array}$$

$$\Rightarrow f(x) = x+3+\frac{-6}{x-2}$$

$\Rightarrow$ Slant asym at $y = x+3$

ver. " " $x = 2$

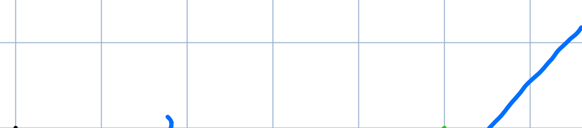
zeros at = $x = 3,\ x = -4$

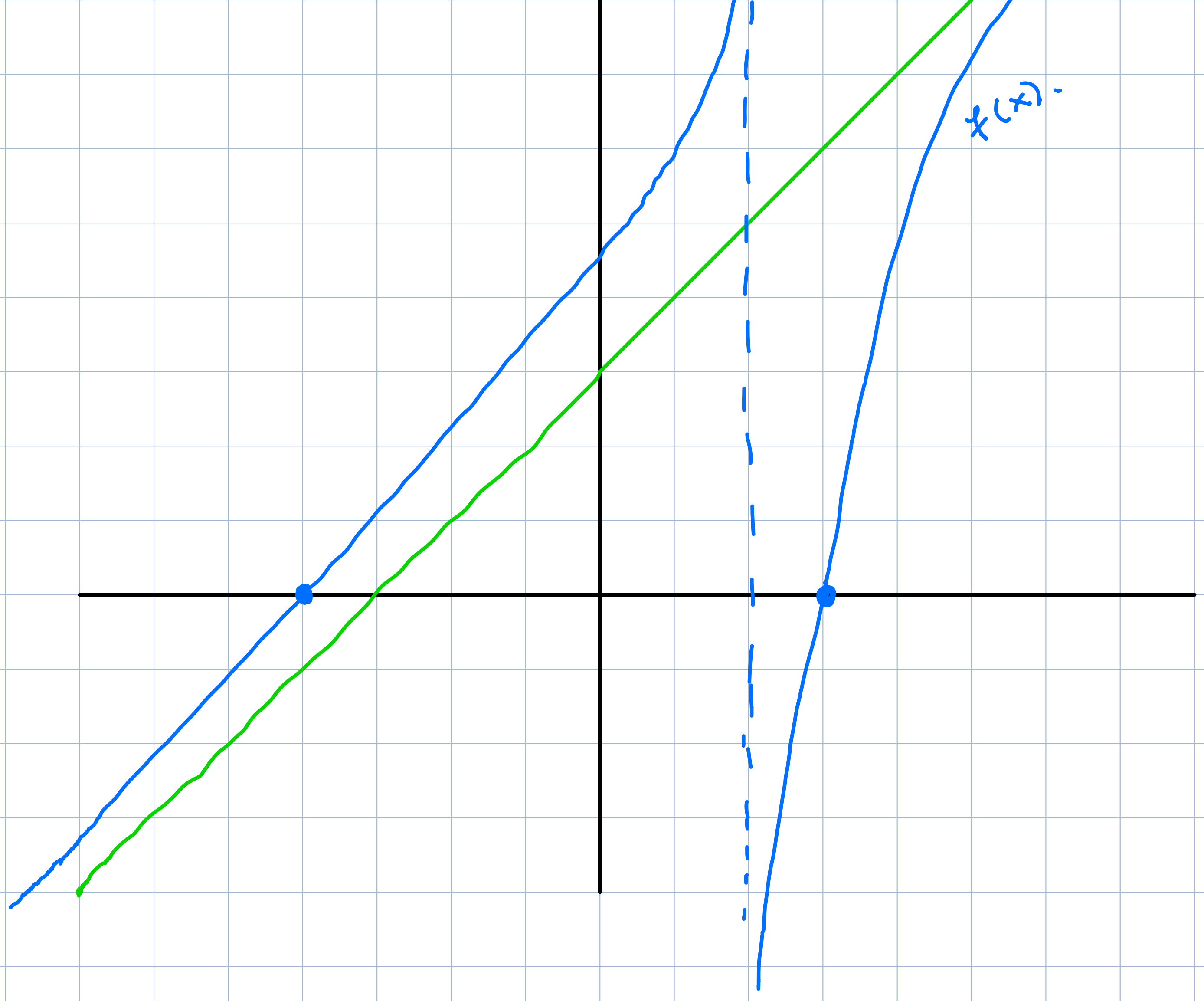
$x<2$ by a bit

$\Rightarrow \dfrac{- \cdot +}{-}$

$x>2$ by a bit

$\dfrac{- \cdot +}{+}$




f(x).