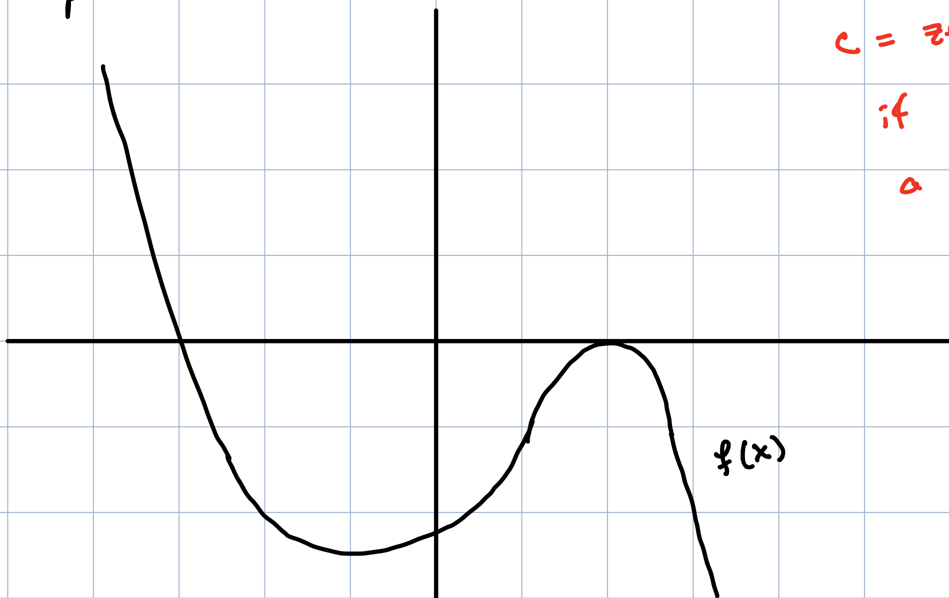
Lecture #13

Warmup: 1) If f is a polynomial of degree 5 w/ graph given below, then what are its zero including their multiplicities

c = zero of mult m. if $(x-c)^m$ is a factor of f.



↳ zeros = -3 ⇝ mult. of -3 is 1

b/c f loc. look like const. $(x+3)^1$

2 ⇝ mult. of 2 is either 2 or 4

Graph of f near a zero of mult. m looks locally like constant $\cdot (x-c)^m$.

2) Completely factor $f(x) = x^3 - 6x + 4$

(Hint: Use the rational root test)

↳ $\pm 1, \pm 2, \pm 4$

Brute force $f(2) = 0$

$\frac{p}{q}$ where

p is factor of constant

q is factor of leading term.

$$
\begin{array}{r|l}
 & x^2 + 2x - 2 \\
\hline
x - 2 & x^3 - 6x + 4 \\
 & -(x^3 - 2x^2) \\
\hline
 & 2x^2 - 6x + 4 \\
 & -(2x^2 - 4x) \\
\hline
 & -2x + 4 \\
 & -(-2x + 4) \\
\hline
 & 0
\end{array}
$$

$$\Rightarrow \quad f = (x-2)\underline{(x^2 + 2x - 2)}$$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2}$$

$$= -2 \pm \sqrt{3} \cdot \sqrt{4} \,/\, 2$$

$$= -1 \pm \sqrt{3}$$

$$\Rightarrow \quad f = (x-2)(x + 1 - \sqrt{3})(x + 1 + \sqrt{3})$$

Recall:

- $i = \sqrt{-1}$, $i^2 = -1$
- cpx # was $a + ib$ where a, b are real #s

Rmk: $(a_0 + ib_0) + (a_1 + ib_1) = (a_0 + a_1) + i(b_0 + b_1)$

$$\Leftrightarrow \quad (1 + i) + (2 + 3i) = 3 + 4i$$

$$(1 - i) + (41 + i) = 42$$

Rmk: $(a_0 + ib_0)(a_1 + ib_1)$

$$= a_0 \cdot a_1 + (a_0 \cdot b_1 + b_0 \cdot a_1) \cdot i + (b_0 \cdot b_1 \cdot i \cdot i)$$

$$= (a_0 \cdot a_1 - b_0 b_1) + (a_0 \cdot b_1 + b_0 \cdot a_1) \cdot i$$

$\hookrightarrow \quad (i)(3+2i) = 3i + 2i^2 = -2+3i$

$\hookrightarrow \quad (2+2i)(2-2i) = 4 - \cancel{4i} + \cancel{4i} - 4i^2 = 8.$

Defn: The cpx conjugate of $a+ib$ is $a-ib = \overline{(a+ib)}$

$\hookrightarrow$ change sign of imaginary part.

$\hookrightarrow \quad \overline{(1+i)} = 1-i$

$\hookrightarrow \quad \overline{(2+i/{-77})} = 2+i/77.$

Fact 8 $\quad (a+ib)(a-ib) = (a+ib)\overline{(a+ib)} = a^2+b^2$

Defn: $\quad \dfrac{a+ib}{c+id} \cdot \dfrac{\overset{= c-id}{\overline{(c+id)}}}{\overline{(c+id)}} = \dfrac{(a+ib)\overline{(c+id)}}{c^2+d^2}$

$\hookrightarrow \quad \dfrac{1+i}{2+i} \cdot \dfrac{(2-i)}{(2-i)} = \dfrac{2-i+2i-i^2}{4+1} = \dfrac{3+i}{5}$

Defn:

- For $r>0$, set $\sqrt{-r} = \sqrt{-1}\cdot\sqrt{r} = i\cdot\sqrt{r}$

  this is called the principle square root of r

- $(\sqrt{-r})^2 = (i\cdot\sqrt{r})^2 = i^2\cdot\sqrt{r}^2 = -1\cdot r = -r$

Warning: $\quad \sqrt{-2}\cdot\sqrt{-3} \neq \sqrt{(-2)\cdot(-3)}$

$\hookrightarrow \quad LHS = i\cdot\sqrt{2}\cdot i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}$

$RHS = \sqrt{6}$

Rmk: f = quad poly, discriminant is < 0, then f has no real soln, but it has cpx solution.

Ex: $f(x) = x^2 + 1$

$$x = \frac{0 \pm \sqrt{0 - 4(1)(1)}}{2} = \frac{\pm\sqrt{-4}}{2} = \pm\frac{i\sqrt{4}}{2} = \pm i.$$

$\Rightarrow$ $\pm i$ are the zeros of f, $f(i) = (i)^2 + 1 = -1 + 1 = 0$.

Section 3.5: Cpx Zeros and the Fund. Thm of Alg.

Rmk: We can make sense of poly. w/ cpx coeffs.

$$f(x) = a_n x^n + \dots + a_1 x^1 + a_0$$

where each a_i is a cpx #, and where they eat and spit out cpx #s.

Thm: • Every $\overset{\deg = n}{\text{poly}}$ w/ cpx coeff has n roots that are possibly cpx #s when counted w/ multiplicity.

$\rightarrow$ $(x-c)^m$ is a factor, then "count" c has a zero m times.

$\hookrightarrow$ $f(x) = x^{77}$, roots 0 w/ mult. 77.

$f(x) = 0 = x^{77} \Rightarrow x = 0$.

↳ Every poly can written as

$$P(x) = a \cdot (x - c_1)(x - c_2) \cdots (x - c_n)$$

where c_i are poss. non-distinct.

Ex: $P(x) = 3x^5 + 24x^3 + 48x$, find the zeros.

$= x(3x^4 + 24x^2 + 48)$

$= 3x(x^4 + 8x^2 + 16)$ $\quad y = x^2$. $\quad x = 0$ is a root.

$= 3x(y^2 + 8y + 16)$

$= 3x(y+4)^2$ $\quad x = \pm 1$

$= 3x(x^2+4)^2$ $\quad x^2 - 1 = 0$

↳ need to know when $x^2 + 4 = 0$

↳ $x^2 = -4 \Rightarrow x = \pm\sqrt{-4} = \pm 2i$

$= 3x(x - 2i)^2(x + 2i)^2$

$\Rightarrow$ zeros are $0, \pm 2i$ w/ $\pm 2i$ each having mult. $= 2$.

Ex: Find poly w/ zeros $x = \pm 2, 3$ w/ mult. $2, 2, 3$ respectively and w/ $P(0) = 5$

FTA $\Rightarrow P(x) = a \cdot (x-2)^2(x+2)^2(x-3)^3$

To pin down a, we look at $P(0) = 5$

$5 = P(0) = a \cdot (-2)^2(2)^2(-3)^3 = a \cdot 16 \cdot (-27)$

$\Rightarrow a = -5/(16 \cdot 27)$.

$$\Rightarrow P(x) = \frac{-5}{16 \cdot 27}(x-2)^2(x+2)^2(x-3)^3$$

Thm: If P has real coeff and z is a zero of P (poss. cpx), then $\bar{z}$ is also a zero of P.

↪ we call the pair z and $\bar{z}$ conjugate roots.

Ex: Find poly of deg $= 3$ w/ real coeff and zeros $x = 1$, $x = 1 + 3i$.

$\Rightarrow$ $1 - 3i = \overline{(1 + 3i)}$ is also a zero of such a poly

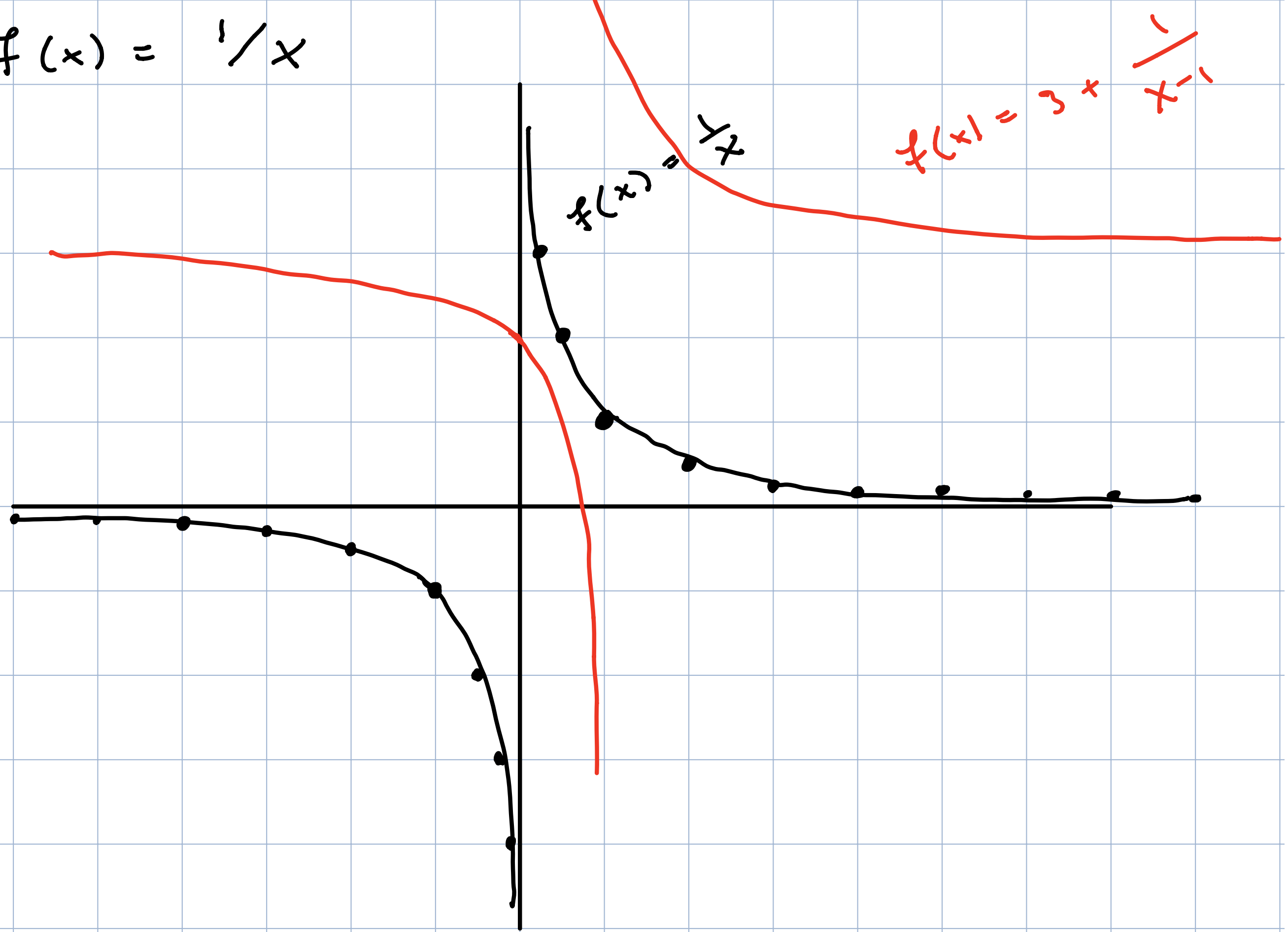
$$\Rightarrow P(x) = (x-1)(x - 1 + 3i)(x - 1 - 3i)$$
$$= (x-1)(x^2 + (-1+3i-1-3i)x + (-1+3i)(-1-3i))$$
$$= (x-1)(x^2 - 2x + (1+9))$$
$$= (x-1)(x^2 - 2x + 10)$$
$$= x^3 + \dots \quad \text{etc.}$$

Section 3.6: Rat'l Function.

Defn: Rat'l fcn is $f(x) = P(x)/Q(x)$ where P, Q are poly. and they have no common factor

↪ ie, $\dfrac{(x-1)x}{(x-1)} \neq$ rat'l fcn

Ex: $f(x) = 1/x$



Ex: $f(x) = \dfrac{ax+b}{cx+d}$ by transformation rules

$$f(x) = \frac{3x+2}{x-1} \overset{\text{Long div.}}{=} a + \frac{b}{x-1}$$

$$\begin{array}{r|l} & 3 \\ \hline x-1 & 3x+2 \\ & -(3x-3) \\ \hline & 5 \end{array}$$

Long div. $\Rightarrow f = 3 + \dfrac{5}{x-1}$

Ex: $f(x) = 3 + \dfrac{1}{x-1}$

Remainder $P(x)/x-c$ is a constant and equal to $P(c)$

Ex: $c = 2$, $P(x) = 3x^2 - 5x + 2$

$$
\begin{array}{r|l}
 & 3x + 1 \\
\hline
x-2 & 3x^2 - 5x + 2 \\
 & -(3x^2 - 6x) \\
\hline
 & x + 2 \\
 & -(x - 2) \\
\hline
 & 4
\end{array}
$$

$\Rightarrow$ remainder $= 4 = P(2)$

$P(2) = 3(2)^2 - 5(2) + 2 = 12 - 10 + 2 = 4$ ✓

Compute $P(c)$ using Long division.