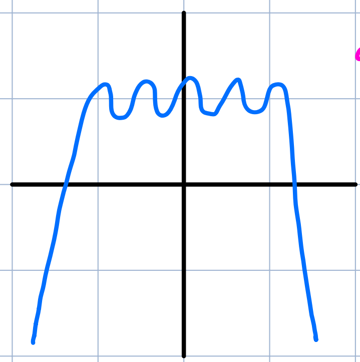
Lecture # 12

Warm-up: What is the minimal degree n such that the drawn graph could be the graph of a polynomial of degree n?



deg ≈ 10

$\Leftrightarrow$ # of real zeros $\leq$ deg

# of extrema $\leq$ deg $- 1$

Section 3.3: Dividing Polynomials

Ex:

$$\frac{37}{8} = 4 + \frac{5}{8}$$

dividend (37), divisor (8), quotient (4), remainder (5)

$37 = 4 \cdot 8 + 5$

Fact: Given two poly $P(x), D(x) \neq 0$, there exist (unique) polys $R(x), Q(x)$ st

i) $\deg(R) < \deg(D)$

ii) $$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

label these guys like above.

$$\Longleftrightarrow$$

$$P(x) = D(x) \cdot Q(x) + R(x)$$

Ex:

$$\frac{x^2-2x+1}{x}$$

$$\begin{array}{r|l} & x \; -2 \\ \hline x & x^2-2x+1 \\ & -(x^2) \\ \hline & -2x+1 \\ & -(-2x) \\ \hline & 1 \end{array}$$

$$\begin{array}{r|l} & 205 \quad R\ 2 \\ \hline 3 & 617 \\ & -600 \\ \hline & 17 \\ & -\ 0 \\ \hline & 17 \\ & 15 \\ \hline & 2 \end{array}$$

$\Rightarrow \quad Q(x) = x-2, \quad R(x) = 1$

$x \cdot (x-2) + 1 = x^2 - 2x + 1 = P(x)$!

Ex:

$$\begin{array}{r|l} & 2x - \frac{1}{3} \\ \hline 3x & 6x^2 - x + 2 \\ & -(6x^2) \\ \hline & -x+2 \\ & -(-x) \\ \hline & 2 \end{array}$$

$\Rightarrow \quad Q(x) = 2x - \frac{1}{3}$

$R = 2$

Ex:

$$\begin{array}{r|l} & x^2+2x+1 \\ \hline x-2 & x^3-3x+4 \\ & -(x^3-2x^2) \\ \hline & 2x^2-3x+4 \\ & -(2x^2-4x) \\ \hline & x+4 \\ & -(x-2) \\ \hline \end{array}$$

Reduce deg.

$\Rightarrow \quad Q = x^2 + 2x + 1 \ , \quad R = 6 .$

Ex:

$$
\begin{array}{r|l}
 & 4x^2 + 2x + \\
\hline
2x^2 - x + 2 & 8x^4 + 6x^2 - 3x + 1 \\
 & -(8x^4 - 4x^3 + 8x^2) \\
\hline
 & 4x^3 - 2x^2 - 3x + 1 \\
 & -(4x^3 - 2x^2 + 4x) \\
\hline
 & -7x + 1
\end{array}
$$

$\rightsquigarrow$ lower deg than $2x^2 - x + 2$

$\Rightarrow$ Done

$\Rightarrow \quad Q = 4x^2 + 2x \ , \quad R = -7x + 1$

Fact: If $P(x)$ is divided by $(x-c)$, then $R = P(c)$.

$\hookrightarrow \quad x^2 - 2x + 1 \ / \ x - 2$, then $R = (2)^2 - 2(2) + 1 = 1$

Proof: there exist Q, R st

$$P = Q \cdot (x-c) + R$$

$\deg(R) < \deg(x-c) = 1 \Rightarrow \deg(R) = 0 \Rightarrow R =$ constant

Plug in c to P.

$$R = Q(c) \cdot \underbrace{(c-c)}_{0} + R = P(c)$$

Ex:

$$\begin{array}{r|l} & x \\ \hline x-2 & x^2-2x+1 \\ & -(x^2-2x) \\ \hline & 1 \end{array} \Rightarrow R = 1 = (2)^2 - 2(2) + 1 = 1$$

Fact: $P(c) = 0$ if and only if $(x-c)$ is a factor of P. ($\Longleftrightarrow$)

Proof: $(\Leftarrow)$: $P = (x-c)\cdot f(x) \Rightarrow P(c) = (c-c)\cdot f(c) = 0$

$(\Rightarrow)$: Write $P(x) = Q(x)\cdot(x-c) + R \overset{!}{=} Q(x)\cdot(x-c)$

but remainder thm $\Rightarrow R = P(c) = 0$

Ex: Factor $P(x) = x^3 - 5x^2 + 7x - 3$ $(P(3) = 0)$

$\Rightarrow$ $(x-3)$ divides P w/ remainder 0.

$$\begin{array}{r|l} & x^2 - 2x + 1 \\ \hline x-3 & x^3 - 5x^2 + 7x - 3 \\ & -(x^3 - 3x^2) \\ \hline & -2x^2 + 7x - 3 \\ & -(-2x^2 + 6x) \\ \hline & x - 3 \\ & -(x-3) \\ \hline & 0 \end{array}$$

$\Rightarrow$ $P = (x-3)(x^2-2x+1) = (x-3)(x-1)^2$

Rmk: Synth. division.

Section 3.4: Real Zeros of Poly.

Fact: $f(x) = a_n x^n + a_{n-1} \cdot x^{n-1} + \dots + a_1 x^1 + a_0$

Spse $a_n \neq 0 \neq a_0$, a_i is an integer

↳ Ex: $3x^{77} + 7x^6 - 2x^2 - 5$, good

$\frac{4}{5} x^2 + 7$, bad

$x^2 - 3x$, bad.

Then every rat'l zero of f is of the form p/q, in lowest terms.

where p is a factor of a_0, q is a factor of a_n.

Ex: $f(x) = 3x^{77} + 3x + 4$

$\Rightarrow$ only poss. rat'l zeros are

$\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$.

"Proof" $f(x) = ax^2 + bx + c$

Spse $f(p/q) = 0$

$\Rightarrow 0 = a(p/q)^2 + b(p/q) + c$

$\Rightarrow 0 = ap^2 + bpq + cq^2$

$\Rightarrow q$ divides $bpq + cq^2 = q(bp + cq) = -ap^2$

$\Rightarrow q$ divides a = leading term $\square$

Ex: $f(x) = x^3 - 3x + 2$, factor it!

$\Rightarrow$ poss. rat'l roots $\pm 1, \pm 2$

$+1$ works, -2 works

$\Rightarrow$ Divide f by $(x-1)$, $(x+2)$ to factor.

$$
\begin{array}{r l}
 & x^2 + x - 2 \\
x - 1 & \overline{\big)\, x^3 - 3x + 2} \\
 & \underline{-(x^3 - x^2)} \\
 & x^2 - 3x + 2 \\
 & \underline{-(x^2 - x)} \\
 & -2x + 2 \\
 & \underline{-(-2x + 2)} \\
 & 0
\end{array}
$$

$\Rightarrow f = (x-1)(x^2 + x - 2)$

$$
\begin{array}{r l}
 & x - 1 \\
x + 2 & \overline{\big)\, x^2 + x - 2} \\
 & \underline{-(x^2 + 2x)} \\
 & -x - 2 \\
 & \underline{-(-x - 2)} \\
 & 0
\end{array}
$$

$\Rightarrow f(x) = (x-1)(x+2)(x-1)$.

$68 - 96 + 28$

$96 - 96$ ✓

$$
\begin{array}{r}
{}^{3}16 \\
\underline{6} \\
96
\end{array}
$$

Ex: $f(x) = x^3 - 6x^2 + 7x + 4$, factor this!

$\Rightarrow$ poss. $= \pm 1, \pm 2, \pm 4$

$\Rightarrow$ 4 is a root!

$$
\begin{array}{r|l}
 & x^2 - 2x - 1 \\
\hline
x-4 & x^3 - 6x^2 + 7x + 4 \\
 & -(x^3 - 4x^2) \\
\hline
 & \quad -2x^2 + 7x + 4 \\
 & -(-2x^2 + 8x) \\
\hline
 & \qquad\qquad -x + 4 \\
 & \qquad\quad -(-x + 4) \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

$\Rightarrow f = (x-4)\underline{(x^2 - 2x - 1)}$.

$\rightarrow$ poss ± 1

Reduced to quad $\Rightarrow$ quad formula.

$$x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

$\Rightarrow$ $x^2 - 2x - 1$ has zeros $1 \pm \sqrt{2}$

$\Rightarrow$ $x^2 - 2x - 1$ has factors $(x - (1 + \sqrt{2}))$, $(x - (1 - \sqrt{2}))$

$\Rightarrow$ $f(x) = (x-4)(x - (1 + \sqrt{2}))(x - (1 - \sqrt{2}))$.

Section 1.6: Complex Numbers.

Ex: $0 = x^2 + 1 \Rightarrow -1 = x^2 \Rightarrow x = \pm\sqrt{-1}$

Defn: A complex number is a combination

$$z = a + ib$$

where a, b are real #s and i is some symbol st $i^2 = -1$ (think $i = \sqrt{-1}$).

↳ Real part of z, $Re(z) = a$

↳ Imaginary " " ", $Im(z) = b$

↳ if $Re(z) = 0$, then z is said to be pure imaginary.

Ex: i is a root of $x^2 + 1 = 0$

$(i)^2 + 1 = -1 + 1 = 0$ ☺