Lecture #10

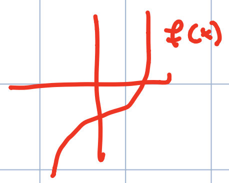
Warm-up: 1) Determine whether or not the functions are 1-to-1.

a) $f(x) = x^3 - 8$

b) $f(x) = x^2 - 2x$

fcn is 1-to-1 if pass hor. line test.

Fcn sends each real # to a # that is unique.

↳ 

⇝ hor. line test ⇒ yes

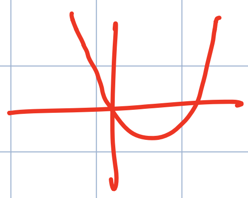
$f(x_0) = f(x_1)$

$\Rightarrow \quad x_0^3 - 8 = x_1^3 - 8$

$\Rightarrow \quad x_0^3 = x_1^3$

$\Rightarrow \quad x_0 = x_1$

$\Rightarrow$ f is one-to-one

↳ 

⇝ hor. line test fails ⇒ no.

$f(2) = (2)^2 - 2(2) = 4 - 4 = 0$

$f(0) = 0^2 - 2(0) = 0 - 0 = 0$

$\Rightarrow$ not 1-to-1.

2) Express $F(x) = \sqrt[3]{x^2 - 3}$ as a composition of functions.

$\hookrightarrow \; f \circ g(x) = f(g(x))$

$G(x) = (x-2)^2 = f \circ g(x)$

where $f(x) = x^2, \; g(x) = x - 2$

$\hookrightarrow \; f(g(x)) = f(x-2) = (x-2)^2 = G(x)$

$\hookrightarrow \; g(x) = x^2 - 3, \; f(x) = \sqrt[3]{x}$

$f \circ g(x) = f(x^2 - 3) = \sqrt[3]{x^2 - 3}.$

$h(x) = x^2, \; K(x) = x - 3$

$K \circ h(x) = K(x^2) = x^2 - 3 = g(x)$

$F(x) = f \circ K \circ h(x).$

Defn: $f : A \longrightarrow B$ (A, B sets of real #), f is 1-to-1 then $f^{-1} : B \longrightarrow A$ is given by

$f^{-1}(y) = x$ where $f(x) = y.$

$\hookrightarrow \; f^{-1}$ is called the inverse of f

$\hookrightarrow \; f \circ f^{-1}(y) = y$

$\hookrightarrow \; f^{-1} \circ f(x) = x$

$\hookrightarrow \; \mathrm{dom}(f^{-1}) = \mathrm{range}(f)$

$\mathrm{range}(f^{-1}) = \mathrm{dom}(f)$

Ex: $f(x) = x - 1, \; f^{-1}(y) = y + 1$

Verify: $f \circ f^{-1}(y) = f(y+1) = y + 1 - 1 = y$

$f^{-1} \circ f(x) = f^{-1}(x-1) = x - 1 + 1 = x$

Fact: If $f \circ g(y) = y$ and $g \circ f(x) = x$, then g is the inverse of f (f is the inv. of g).

Ex: $f(x) = x^3$, $f^{-1}(y) = \sqrt[3]{y}$

↳ $f \circ f^{-1}(y) = f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y$

$f^{-1} \circ f(x) =$ sim.

Rmk: Solve for the inverse by hand:

↳ $y = f(x)$

↳ solve for x in terms of y

↳ Set $f^{-1}(y) = x =$ exp. in y.

Ex: $f(x) = x - 1$

$y = f(x) = x - 1$

$\Rightarrow x = y + 1 = f^{-1}(y)$

Ex: $f(x) = (x^5 - 4)/2$

$y = f(x) = (x^5 - 4)/2$

$\Rightarrow 2y = x^5 - 4$

$\Rightarrow 2y + 4 = x^5$

$\Rightarrow \sqrt[5]{2y+4} = x = f^{-1}(y)$

↳ $f \circ f^{-1}(y) = f(\sqrt[5]{2y+4})$

$= ((\sqrt[5]{2y+4})^5 - 4)/2$

$= (2y + 4 - 4)/2$

$= 2y/2$

$= y.$ ☺

Ex: $f(x) = x-4 / x-2$

$$y = f(x) = (x-4)/(x-2)$$

$$\Rightarrow y(x-2) = x-4$$

$$\Rightarrow yx - 2y = x-4$$

$$\Rightarrow -2y+4 = x - yx$$

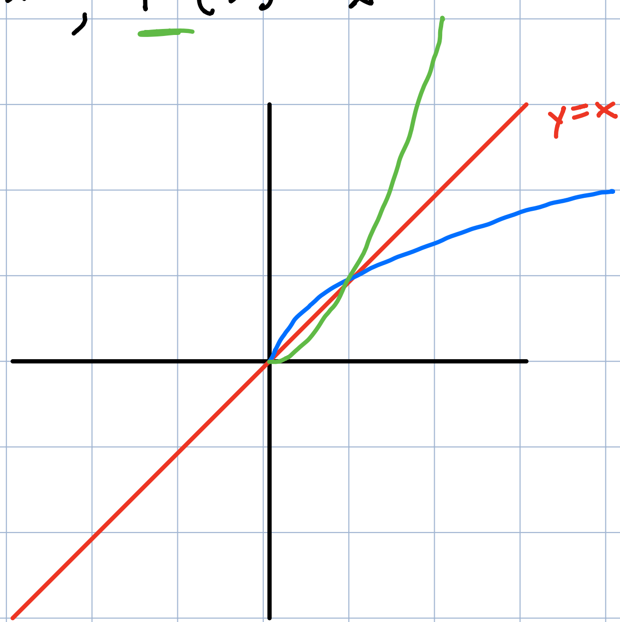
$$\Rightarrow -2y+4 = x(1-y)$$

$$\Rightarrow x = \frac{-2y+4}{1-y} = \frac{2y-4}{y-1} = f^{-1}(y).$$

Rmk: Graph of f^{-1} is the reflection of the graph of f across the line $y=x$

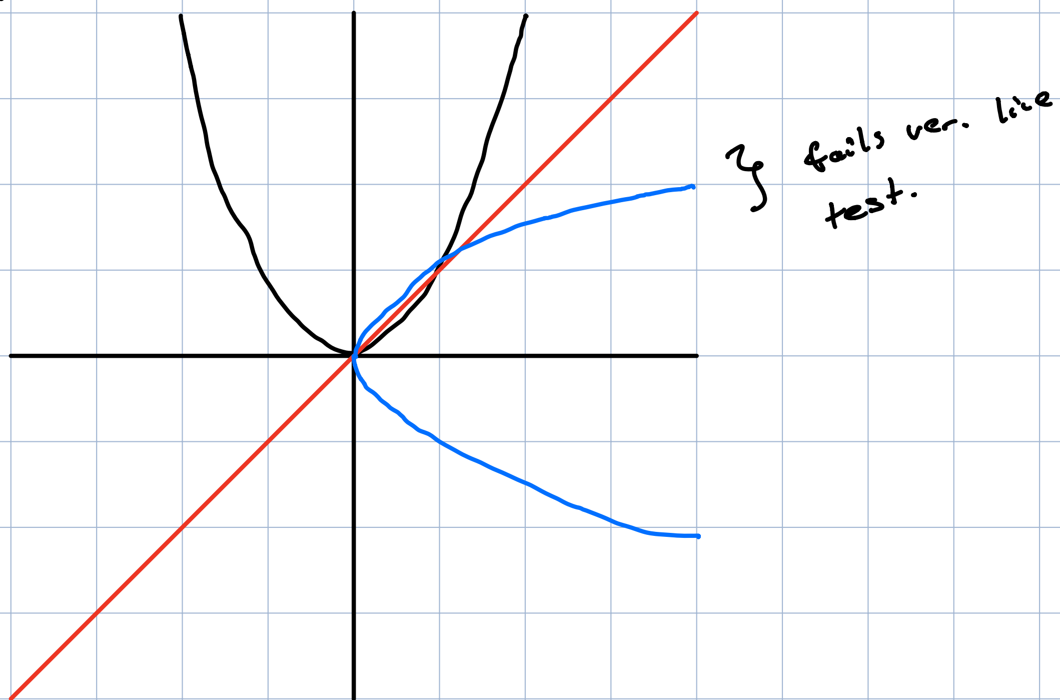
↪ ref. across $y=x$ ⇒ interchange x, y values.

Ex: $f(x) = \sqrt{x}$, $f^{-1}(x) = x^2$



Rmk: Reflect Hor. lines across $y = x$, they go to ver. lines. Fcn pass hor line test $\Rightarrow$ inv fcn passes ver. line test (ie, it actually is a fcn).

Ex: $f(x) = x^2$



Section 3.1: Quadratic Functions (and models)

Defn: The standard form of a quadratic function

$$f(x) = ax^2 + bx + c$$

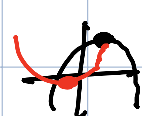
is the form

$$f(x) = a(x-h)^2 + k$$

interchange via completing the square

$\Leftrightarrow f(x) = x^2 + x + \frac{1}{4} - \frac{1}{4} = (x + \frac{1}{2})^2 - \frac{1}{4}$

The vertex of the graph of f is the point (h, k)

$\hookrightarrow$ vertex is the loc. min/max of the quad 

If $a > 0$, then f opens upwards

" $a<0$, " - " downwards

f achieves a local min/max value of k at h ($a>0$ / $a<0$)

Fact: $\text{range}(f) = [k, +\infty)$ when $a>0$
$= (-\infty, k]$ when $a<0$

Ex: $f(x) = 2x^2 - 12x + 13$

a) Std form
b) Vertex
c) x-intercepts
d) y-intercepts
e) graph
f) dom/range

a) $f(x) = 2x^2 - 12x + 13$

$= 2(x^2 - 6x) + 13$

$= 2((x-3)^2 - 9) + 13$

$= 2(x-3)^2 - 18 + 13$

$= 2(x-3)^2 - 5$

$$\begin{array}{r} 26 \\ \times\ 4 \\ \hline 104 \end{array}$$

b) Vertex: $(3, -5)$

c) x-intercepts: $f(x) = 0 = 2x^2 - 12x + 13$

when $x = \dfrac{+12 \pm \sqrt{144 - 4(2)(13)}}{4}$

$= \dfrac{12 \pm \sqrt{40}}{4}$

$= \dfrac{12 \pm \sqrt{4} \cdot \sqrt{10}}{}$

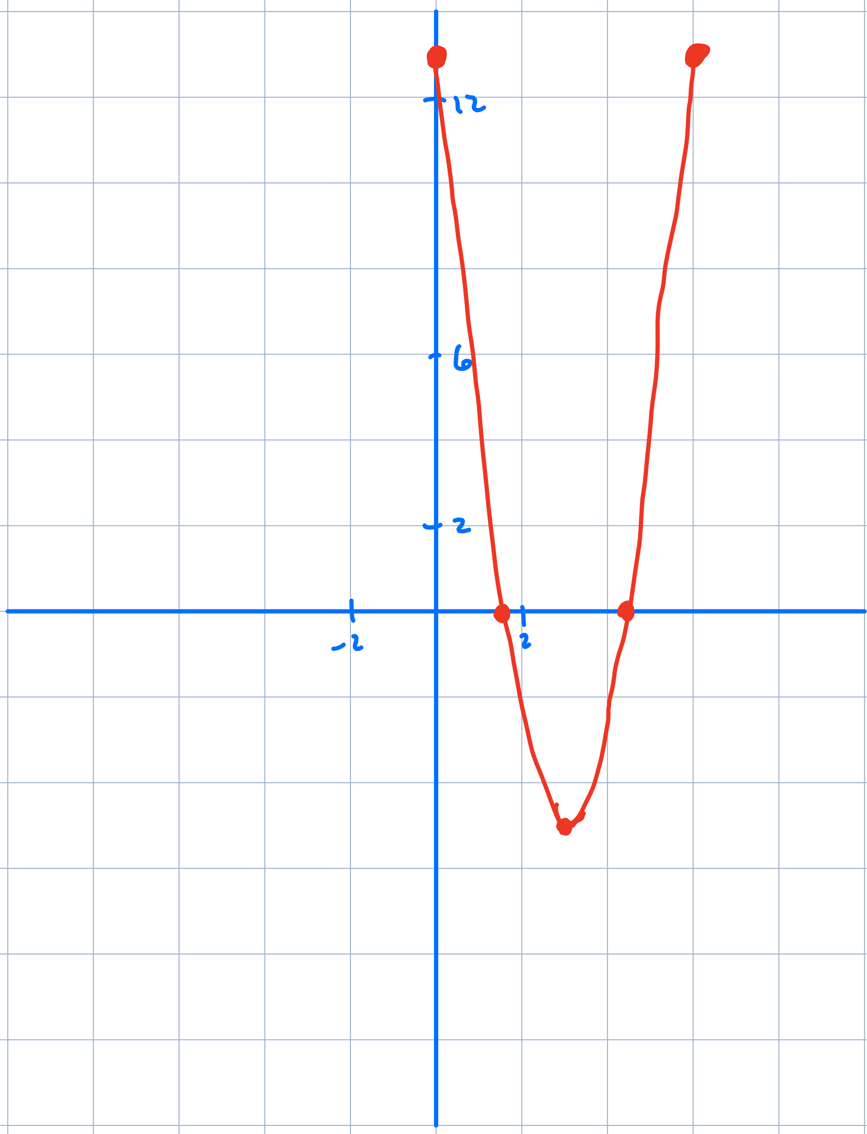
$$= \frac{\qquad}{4}$$

$$= \frac{6 \pm \sqrt{10}}{2} = 3 \pm \frac{\sqrt{10}}{2} \simeq 3 \pm 1.5$$

d) y-intercept: $f(0) = 2(0)^2 - 12(0) + 13 = 13$

$(0, 13)$

e)



f) $\text{dom}(f) = (-\infty, +\infty)$, $\text{range}(f) = [-5, +\infty)$.

range(f) is det by a and vertex

$$\begin{array}{r} 1 \\ 49 \\ 20 \\ \hline 980 \end{array}$$

Ex: $f(x) = -5x^2 + 30x - 49$, discrim(f)

$30^2 - 4(-5)(-49)$

$30^2 - 20 \cdot 49$

$900 - 980 < 0.$

a) Std form

b) vertex

c) range

a) $f(x) = -5x^2 + 30x - 49$

$= -5(x^2 - 6x) - 49$

$= -5(\underline{x^2 - 6x + 9} - 9) - 49$

$= -5(\underline{(x-3)^2} - 9) - 49$

$= -5(x-3)^2 - 4$

b) vertex $= (3, -4)$

c) range$(f) = (-\infty, -4]$