Section 1.1: Real Numbers

Defn: The natural numbers are countable #'s

↪ $0, 1, 2, 3, 4, 5, \ldots, 77, \ldots, 7777, \ldots$

Defn: The integers are the nat. #'s and their negatives

$\ldots, -42, -41, -40, \ldots, -1, 0, 1, 2, 3, \ldots, 77, \ldots$

Defn: A rational # is a ratio of two integers, $\frac{a}{b}$.

↪ $\frac{7}{11}, \frac{2}{3}, \frac{-4}{3}, 4 = \frac{4}{1}$

Defn: A # is irrational if it is not rat'l.

↪ $\pi, \sqrt{2}, \sqrt{7/11}, e \approx 2.72\ldots, \pi^2/4$

Defn: The collection of all num. is the real #s, denote $\mathbb{R}$.

Rmk: Every reall # admits a decimal expansion

↪ $1 = 1.0, \frac{1}{4} = 0.25, \frac{1}{8} = 0.125$

$\frac{1}{6} = 0.1\overline{6666}, \pi = 3.141596\ldots$?

A # is rat'l if its dec. expansion is finite or eventually repeats

Rules: (Commutativity of add/mult)

$\hookrightarrow$ $a+b = b+a$; $2+7 = 9 = 7+2$

$\hookrightarrow$ $a \cdot b = b \cdot a$; $2 \cdot 7 = 14 = 7 \cdot 2$

(Associativity of ")

$\hookrightarrow$ $(a+b)+c = a+(b+c)$;

$(2+3)+4 = 5+4 = 9$ ✓

$2+(3+4) = 2+7 = 9$

$\hookrightarrow$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(Distributive Law)

$\hookrightarrow$ $a(b+c) = a \cdot b + a \cdot c$

$2(3+4) = 2 \cdot 7 = 14$ ✓

$2 \cdot 3 + 2 \cdot 4 = 6+8 = 14$

<u>Ex:</u>

$$
\begin{aligned}
(a+b)(x+y) &= (a+b)x + (a+b)y \\
&= x(a+b) + y(a+b) \\
&= ax + bx + ay + by.
\end{aligned}
$$

Rule:

1) $(-1)a = -a$
2) $-(-a) = a$
3) $(-a) \cdot b = -(ab)$
4) $(-a)(-b) = ab$; $(-2)(-7) = 14$
5) $-(a+b) = -a-b$; $-(2+7) = -9$

$-2-7 = -9$

Ex: $-2(x-y) = (-2)x + (-2)(-y) = -2x + 2y$

Defn: $a \div b := a \cdot \frac{1}{b} = \frac{a}{b}$

Rule:

1) $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$; $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{2 \cdot 4} = \frac{3}{8}$.

2) $\frac{a}{b} \div \frac{c}{d} \overset{\text{reciprocal}}{=} \frac{a}{b} \cdot \frac{d}{c}$.

3) $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

4) $\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \overset{=1}{\frac{d}{d}} + \frac{c}{d} \cdot \overset{=1}{\frac{b}{b}}$

$= \frac{ad}{bd} + \frac{cb}{db}$

$= \frac{ad + cb}{bd}$

5) $\frac{ac}{bc} = \frac{a}{b}$; $\frac{10}{20} = \frac{\cancel{2} \cdot 5}{\cancel{2} \cdot 10} = \frac{5}{10} = \frac{\cancel{5} \cdot 1}{\cancel{5} \cdot 2} = \frac{1}{2}$

6) $\frac{a}{b} = \frac{c}{d} \implies ad = cb$

$\frac{a}{b} \cdot d = \frac{c}{\cancel{d}} \cdot \cancel{d} = c$

$\frac{ad}{\cancel{b}} \cdot \cancel{b} = c \cdot b \implies ad = cb$

$\Leftrightarrow \quad \frac{2}{10} = \frac{1}{5} \quad \Longrightarrow \quad 2 \cdot 5 = 10 = 10 \cdot 1.$

Defn: A fraction p/q is in lowest terms if there is not a $c \neq 1$ st

$$\frac{p}{q} = \frac{ac}{bc}$$

Ex: $\frac{2}{10}$ is not in lowest terms

$$\frac{2}{10} = \frac{2 \cdot 1}{2 \cdot 5} \quad (c = 2).$$

Defn: $a > b$ if and only if $a - b > 0$

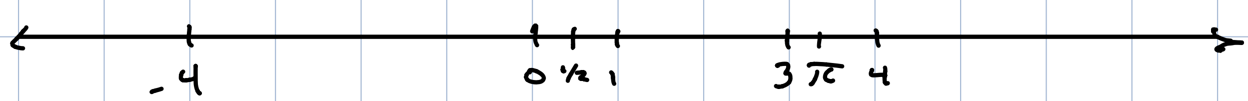
$\Leftrightarrow$ $7 > 2$, $7 - 2 = 5 > 0$

$a \geq b$ " " " " $a - b \geq 0$

If $a > 0$, then a is positive

" $a < 0$, " " " negative.

RmK: $\mathbb{R}$



Defn: A set of numbers is a collection of real numbers

↪ collection could be infinite

↪ Ex: $\mathbb{R}$ is a set

Ex: $\{2\}$

Ex: Numbers less than or equal to 2

Typically, we denote a set w/ $\{\ldots\}$.

↪ Ex: $\{0, 1, 2, 42, 77, 88\}$.

Defn: The union of two sets is the set whose numbers are the numbers in said two sets

↪ Ex: $\{1,2,3,4,5\} \underset{\text{union}}{\cup} \{1,2,42,77,88\}$

$\|$

$\{1,2,3,4,5,42,77,88\}$.

Ex: $\{1,2,3\} \cup \{1,2,4\} = \{1,2,3,4\}$

Ex: $\{1,2\} \cup \{1\} = \{1,2\}$

The intersection of " " is the set " " " the numbers contained in both sets

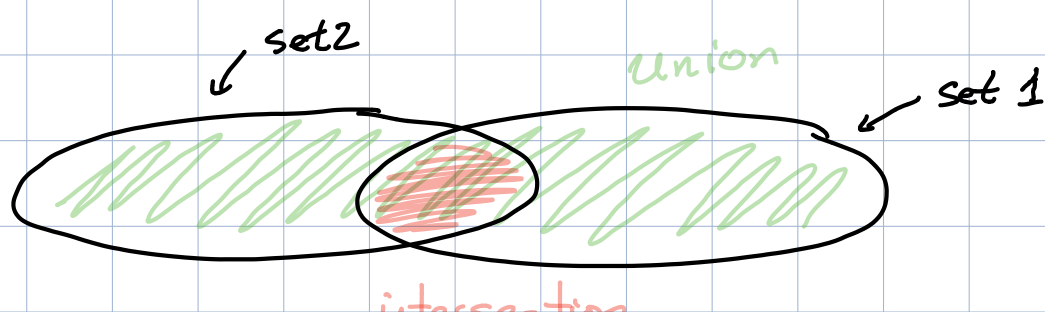
↪ Ex: $\{1,2,3,4,5\} \underset{\text{intersection}}{\cap} \{1,2,42,77,88\}$

$\|$

$\{1,2\}$

Ex: $\{1,2\} \cap \{1\} = \{1\}$

Picture:



Note: Set builder notation

↳ $\{x \mid 0 < x \leq 777\}$

The set of real numbers that satisfy such that x is greater than zero but less than or equal to 777.

77 is in

88 " " "

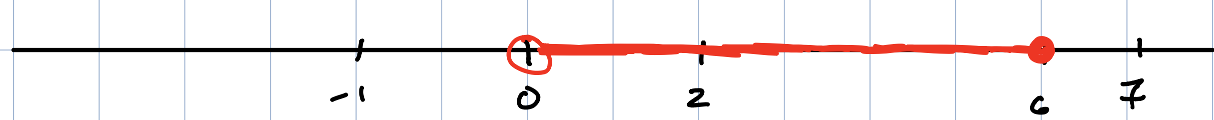
888 is <u>not</u> in this set

Ex: $\{x \mid 0 < x < 7\} \cap \{x \mid -1 < x \leq 6\}$

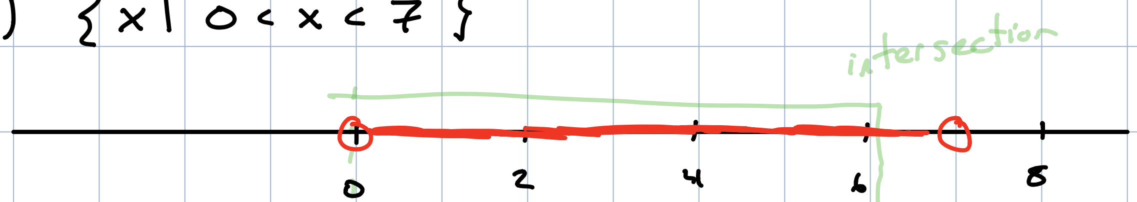
$$= \{x \mid 0 < x \leq 6\}$$

Picture: Graph of a set is the following:

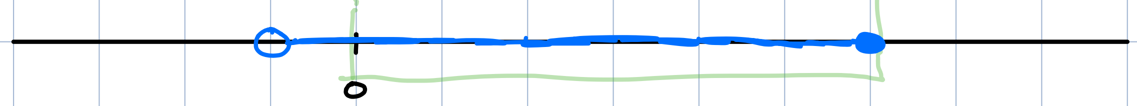
$\{x \mid 0 < x \leq 6\}$



Ex: (cont.) $\{x \mid 0 < x < 7\}$

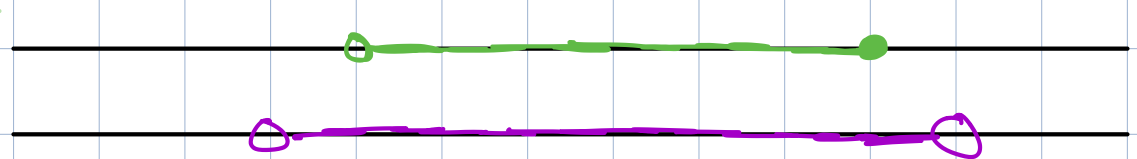


$\{x \mid -1 < x \leq 6\}$



intersection

union



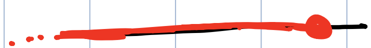
Defn: Interval notation

↳ $(a,b) = \{x \mid a < x < b\}$ 

↳ $[a,b] = \{x \mid a \le x \le b\}$ 

↳ $(a,b] = \{x \mid a < x \le b\}$ 

↳ $[a,b)$ 

↳ $(-\infty, a\overset{)}{]} = \{x \mid x \overset{<}{\le} a\}$ 

↳ $[b, +\infty]$

↳ $(b, +\infty)$

↳ $(-\infty, +\infty) = \mathbb{R}$

Defn: (Absolute value)

$$|a| = \begin{cases} a, & a \ge 0 \\ -a, & a \le 0 \end{cases}$$

↳ $|-2| = -(-2) = 2$

↳ dist of a num. from zero

Rule:

i) $|a| \ge 0$

ii) $|a| = |-a|$; $\overset{2}{|2|} = \overset{''}{|-2|}$

iii) $|a| \cdot |b| = |ab|$; $|-2| \cdot |-5| = 2 \cdot 5 = 10$

iv) $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$ $\qquad |(-2)\cdot(-5)| = |10| = 10$ ✓

v) $|a+b| \le |a| + |b|$

Defn: The dist. from a to b $|a-b|$

↳ dist. $\frac{2}{15}$ to $\frac{11}{20}$

$$\left|\frac{2}{15}-\frac{11}{20}\right| = \left|\frac{20\cdot 2}{15\cdot 20}-\frac{15\cdot 11}{15\cdot 20}\right|$$

$$= \left|\frac{40-166}{15\cdot 20}\right|$$

$$= \left|\frac{-126}{15\cdot 20}\right|$$

$$= \frac{126}{300}$$

$$= \ldots$$

n nat. number

<u>Defn:</u> $a^n = \underbrace{a\cdot \ldots \cdot a}_{n\text{-times}}$; $2^3 = 2\cdot 2\cdot 2 = 8$

$5^2 = 5\cdot 5 = 25$

$(-2)^5 = -32$

$\hookrightarrow$ a is called the base

n is called the exponent

<u>Ex:</u> $\left(\frac{1}{4}\right)^2 = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$

<u>Defn:</u> $a^{-n} = \frac{1}{a^n}$; $a^0 \equiv 1$ $(a \neq 0)$.

<u>Ex:</u> $\left(\frac{1}{4}\right)^{-2} = \frac{1}{(1/4)^2} = 4^2 = 16$

$a^{-1} = \frac{1}{a}$