Lecture # 9

Title: More change of variables + Vector fields

Section: Stewart 15.9, 16.1

Remark:

- $$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

  where $x(u) = x$, $x(c) = a$, $x(d) = b$

- If $u(x) = u$, $u(a) = c$, $u(b) = d$, then

  $$\int_a^b f(u(x))\frac{du}{dx}\,dx = \int_c^d f(u)\,du$$

- These are equivalent! We just changed the roles of x and u.

Ex:

$$\int_0^{\pi/2} \sin^2(x)\cos(x)\,dx = \int_0^1 u^2\,du$$

$u = \sin(x)$, $du = \cos(x)\,dx$

Theorem: Spse $T(u,v) = (x(u,v), y(u,v))$

Spse The image of S under T is R.

$$\Rightarrow \iint_R f(x,y)\, dA$$

$$= \iint_S f(x(u,v), y(u,v)) \cdot |J(T)|\, du\, dv$$

where

$$J(T) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

* partial derivatives wrt u, v.

Notn: $T(u,v) = (x(u,v), y(u,v)) = (x,y)$

$T^{-1}(x,y) = (u(x,y), v(x,y)) = (u,v)$

Fact:

$$\iint_R f(u(x,y), v(x,y))\, |J(T^{-1})|\, dA$$

$$= \iint_S f(u,v)\, du\, dv$$

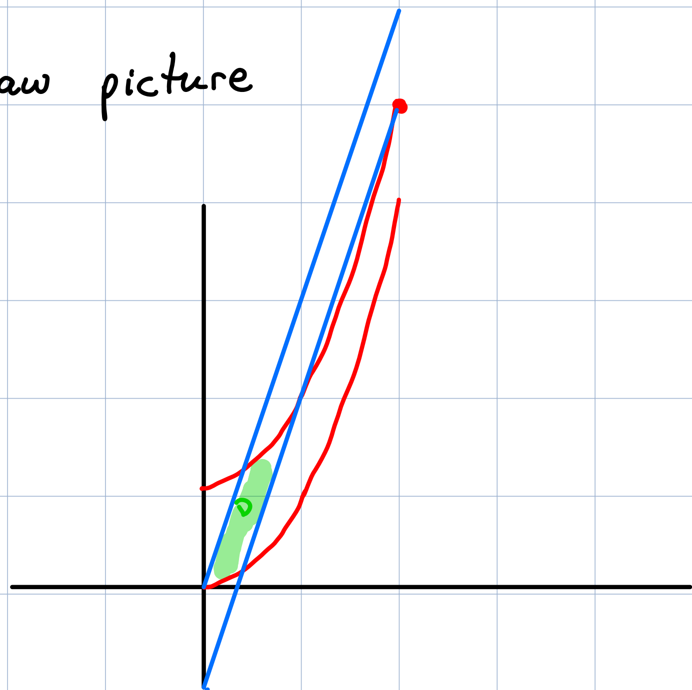
where

$$J(T^{-1}) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Fact: $J(T^{-1})(x(u,v), y(u,v)) = J(T)(u,v)$

Example: Compute $\iint_R \sin(y-x^2)(-2x+3)\,dA$ over the region bounded by $y=x^2$, $y=x^2+1$, $y=3x$, $y=3x-1$.

Soln: ⓪ Draw picture



① Pick substitutions: $u = y - x^2$, $v = y - 3x$

② Find new bounds: $0 \le u \le 1$, $-1 \le v \le 0$

③ Compute Jacobian:

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -2x & 1 \\ -3 & 1 \end{vmatrix} = -2x + 3$$

④ Apply change of variables formula.

$$\iint_R f(x,y)\, dA = \iint_R \sin\big(\overset{u}{y - x^2}\big) \cdot \overset{|J(T^{-1})|\, dA}{(-2x+3)\, dA}$$

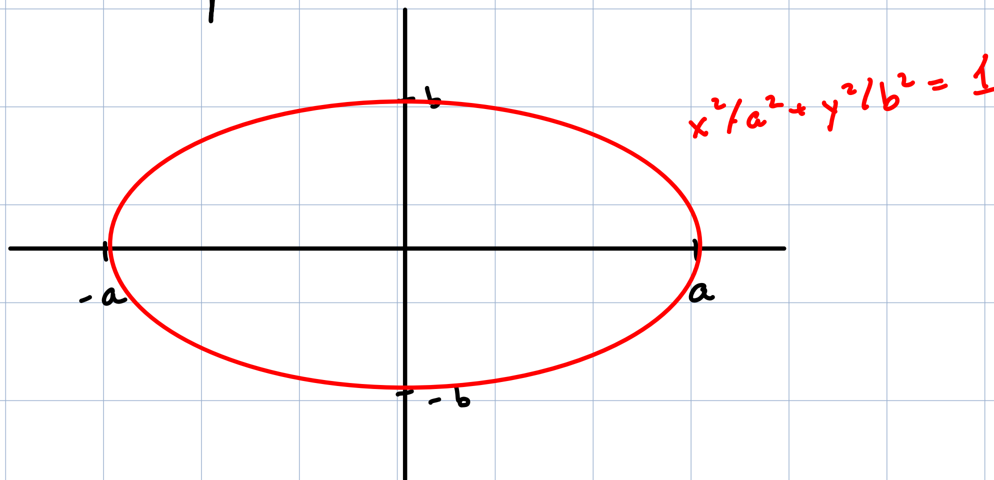
$$= \iint_S \sin(u)\, du\, dv$$

$$= \int_{-1}^{0} \int_{0}^{1} \sin(u)\, du\, dv$$

$$= -\cos(1) + 1$$

Example: Compute $\iint_D dA$ where D is region enclosed by $x^2/a^2 + y^2/b^2 = 1$.

Soln: ⓪ Draw picture



① fix transformation: $x/a = u$, $y/b = v$

$\Rightarrow T(u,v) = (au, bv) = (x,y)$

② So for new region

$$x^2/a^2 + y^2/b^2 = 1 \iff u^2 + v^2 = 1$$

③ Compute Jacobian

$$J(T) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

④ Set up integral and solve

$$\iint_D dA = \iint_S |ab| \, du\,dv$$

$$= \int_0^{2\pi} \int_0^1 r \, |ab| \, dr\,d\theta$$

$u = r\cos\theta$

$v = r\sin\theta$

$$= ab \cdot \pi$$

Review:
- A vector $\mathbb{R}^3$ is a triple (a, b, c) where a, b, c are constants
- Sometimes write

$$(a, b, c) = a \cdot \vec{i} + b \cdot \vec{j} + c \cdot \vec{k}$$

↳ coeff on $\vec{i}$ = x-direction contribution

" " $\vec{j}$ = y- " "

" " $\vec{k}$ = z " "

↳ these are the components of the vectors.

- A vector gives a direction and a magnitude

Remark: Vector in $\mathbb{R}^2$ is just a vector w/ $\vec{k}$ component zero.

↳ $\vec{v} = v_1 \vec{i} + v_2 \vec{j} = (v_1, v_2)$

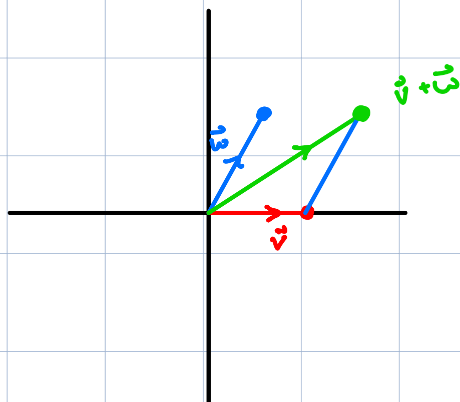
Vector algebra

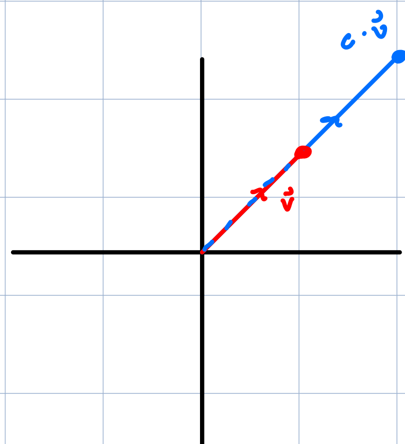
Notn: Let $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$ be two vectors.

Remark: Addition: $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

Scaling: $c \cdot \vec{v} = (c \cdot v_1, c \cdot v_2, c \cdot v_3)$

Norm: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$





Defn: Dot product: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

Fact: $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos(\theta)$ where θ is angle between $\vec{v}$ and $\vec{w}$.

↳ $\vec{v} \perp \vec{w} \Rightarrow \theta = \pi/2 \Rightarrow \vec{v} \cdot \vec{w} = 0$

Defn: *Cross product:

$$\vec{v} \times \vec{w} = (v_2 \cdot w_3 - v_3 \cdot w_2,\ v_3 \cdot w_1 - v_1 \cdot w_3,\ v_1 w_2 - v_2 \cdot w_1)$$

Remark:

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix}$$

* for vectors in $\mathbb{R}^3$.

Fact:

- $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$
- $|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin(\theta)$

  ↳ If $\vec{v}, \vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$

Fact:

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (a \cdot c) b - (a \cdot b) c$$

Vector Fields

Defnː A vector field on $\mathbb{R}^2$ is a function F that assigns to each point (x, y) a vector in $\mathbb{R}^2$.

Ideaː To each point, we associate a direction and magnitude.

Exampleː 1) Wind blowing over surface

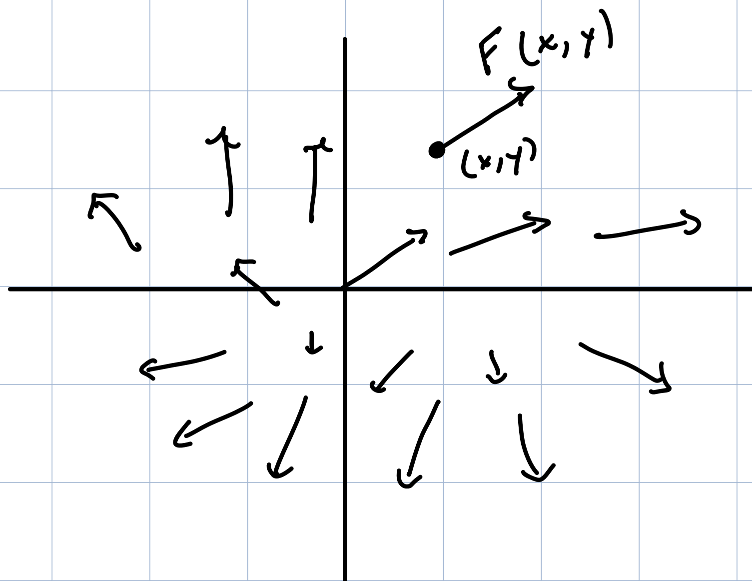
2) Force in electromagnetic/gravitational field.

Remark: To each (x,y), $F(x,y)$ have $\vec{i}$ component and $\vec{j}$ component

$$\Rightarrow F(x,y) = P(x,y)\,\vec{i} + Q(x,y)\cdot\vec{j} = (P(x,y), Q(x,y))$$

where P, Q are functions on $\mathbb{R}^2$.

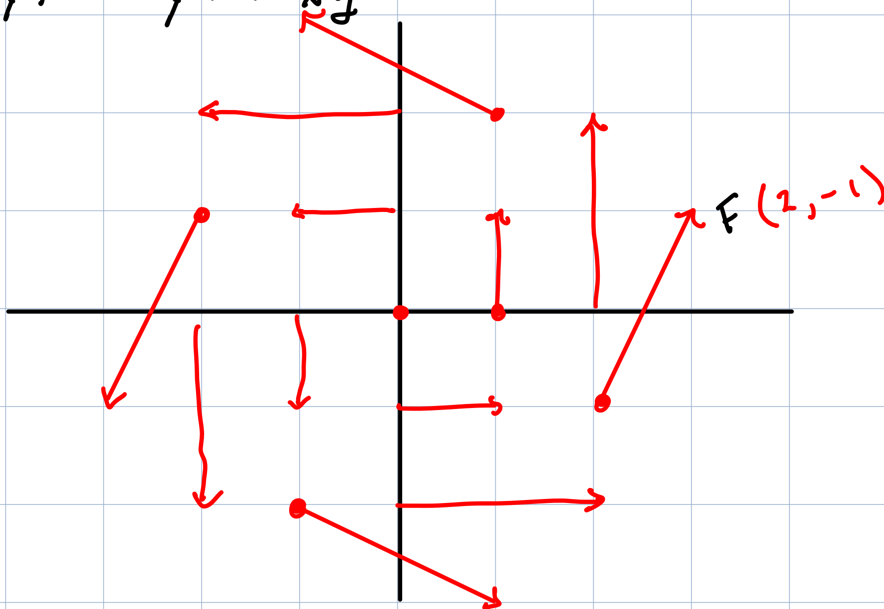
Example:



Defn: A vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x,y,z) a vector in $\mathbb{R}^3$.

$\hookrightarrow F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

Example: $F(x,y) = -y\vec{i} + x\vec{j}$



Example:

Figure 7

$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

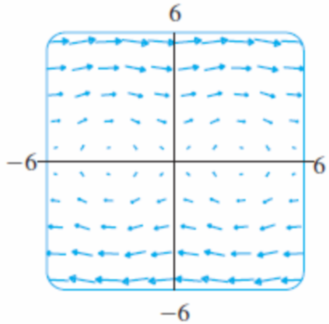
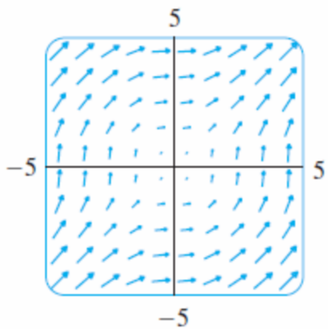


Figure 8

$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$



Notn: Spse C is a plane curve given by the parametric equation: $x = x(t)$, $y = y(t)$

Often think of position as vector, write

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$$

for this curve

Ex: C = unit circle

↳ $x(t) = \cos(t)$, $y(t) = \sin(t)$

C = graph of fcn $f(x)$

↳ $x(t) = t$, $y(t) = f(t)$

C = crazier example

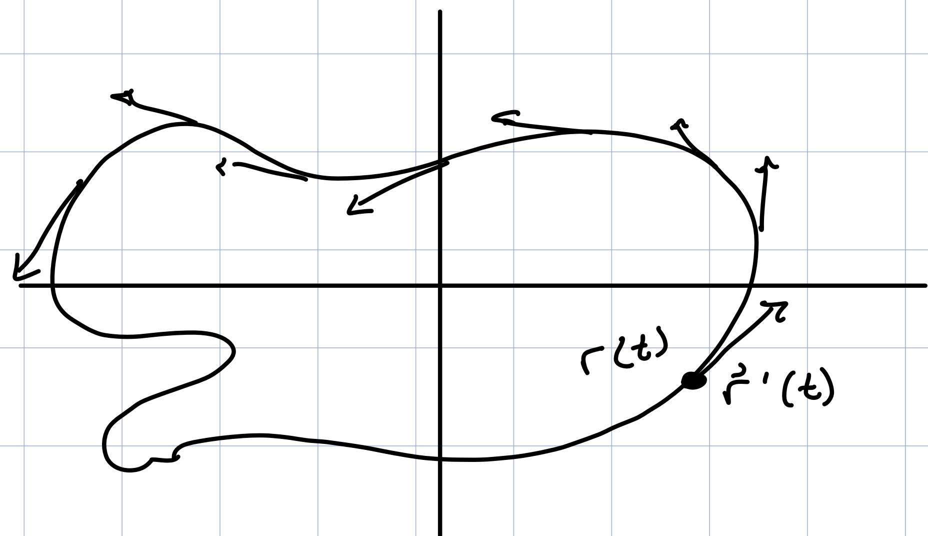
↳ $x(t) = \sin(2t) + 3\sin(t)$, $y(t) = 2\sin(3t)$

Defn: The velocity of a parametric curve $\vec{r}(t)$ is the vector

$$\vec{r}\,'(t) = \frac{dx}{dt}(t)\cdot\vec{i} + \frac{dy}{dt}(t)\,\vec{j}$$

↳ This is tangent to the curve

Picture:



Ex:

1) $x(t) = \cos(t)$, $y(t) = \sin(t)$

$\Rightarrow \vec{r}\,'(t) = -\sin(t)\,\vec{i} + \cos(t)\,\vec{j}$

2) $x(t) = t$, $y(t) = f(t)$

$\Rightarrow \vec{r}\,'(t) = \vec{i} + f'(t)\,\vec{j}$

3) $x(t) = \sin(2t) + 3\sin(t)$, $y(t) = 2\sin(3t)$

$\Rightarrow \vec{r}\,'(t) = \left(2\cos(2t) + 3\cos(t)\right)\vec{i} + 6\cos(3t)\,\vec{j}$

Defn: A parametric curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ is a flow line for a vector field F if

$$F(r(t)) = F(x(t), y(t)) = \frac{dx}{dt}(t)\,\vec{i} + \frac{dy}{dt}(t)\,\vec{j} = \vec{r}\,'(t)$$

↳ Velocity of curve at $x(t), y(t)$ agrees w/ vector from vector field F at $x(t), y(t)$.

↳ paths traced out by particles being push by the vector field.

Figure 7

$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

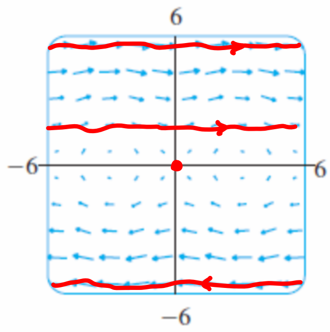
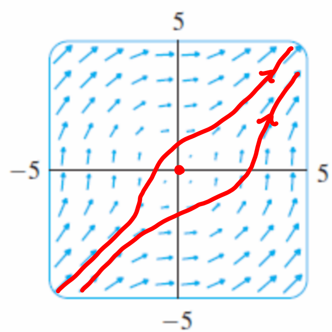


Figure 8

$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

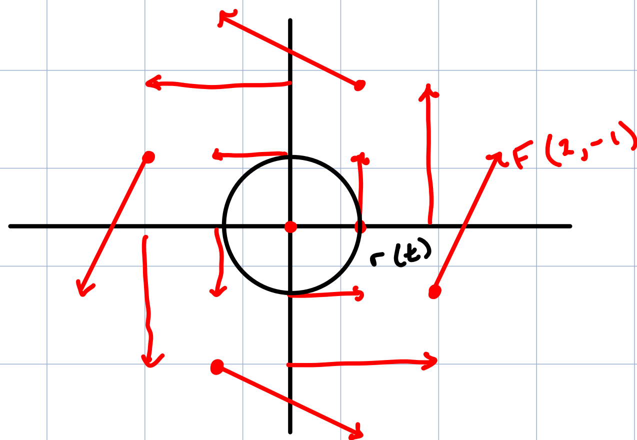


Example: $F = -y\,\vec{i} + x\cdot\vec{j}$, $\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j}$

$\hookrightarrow F(x(t), y(t)) = -\sin(t)\cdot\vec{i} + \cos(t)\cdot\vec{j}$

$= x'(t)\,\vec{i} + y'(t)\,\vec{j}.$

$\Rightarrow$ it is a flowline



Example: $F = x\cdot\vec{i} + 2y\cdot\vec{j}$, $\vec{r}(t) = e^{t}\,\vec{i} + e^{2t}\,\vec{j}$

$\hookrightarrow F(x(t), y(t)) = e^{t}\,\vec{i} + 2e^{2t}\,\vec{j} = x'(t)\,\vec{i} + y'(t)\,\vec{j}$