Lecture # 8

Title: Change of variables in multiple integrals

Section: Stewart 15.9

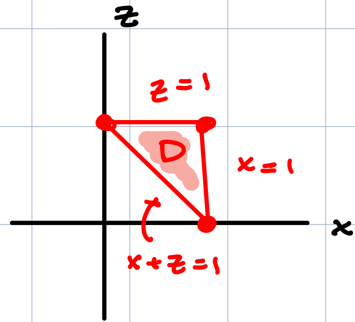
Warm-up: ① Compute the volume of $x^2 = y$ over the triangle in the xz-plane w/ vertices $(0,1)$, $(1,0)$, $(1,1)$.

② **Question 1.3** (5 points) *Set up a double integral using polar coordinates that computes the volume of the solid that is bounded above by* $z = -4 + \sqrt{x^2 + y^2}$, *bounded below* $z = 0$, *and is inside of* $x^2 + y^2 = 36$. *Provide a sketch to justify your answer. (Do not evaluate your double integral.)*

↳ Use a triple integral w/ cylindrical coordinates

Soln:

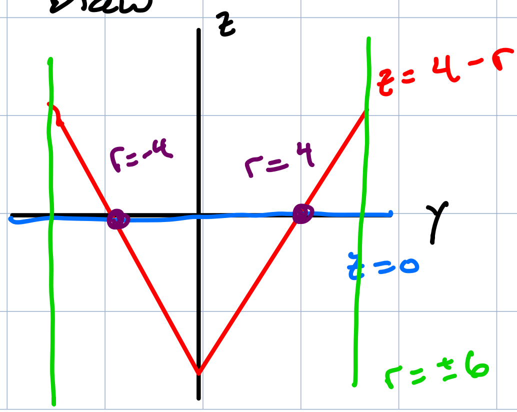
① • Draw D



• Set up integral w/ bounds:

$$\int_0^1 \int_{1-z}^1 \int_0^{x^2} dy\,dx\,dz = \int_0^1 \int_{1-z}^1 x^2 \,dx\,dz = \text{etc.}$$

② • Draw



• Integral

$$\int_0^{2\pi} \int_4^6 \int_0^{4-r} r \,dz\,dr\,d\theta$$

Review:

- $$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

  where $x(u) = x$, $x(c) = a$, $x(d) = b$

  ↳ x as a function of u.

- So $x(u) : [c, d] \longrightarrow [a, b]$ transforms an interval into another (some "stretching / shifting")

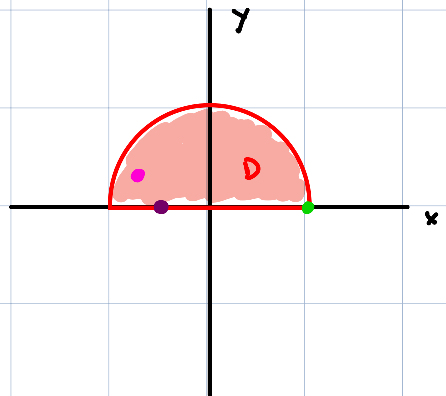
- Idea: Undo the chain rule.

- Find a better looking system of coordinates that makes the integral easier!

Motivation: Polar integrals: $x = r\cos(\theta)$, $y = r\sin(\theta)$

$$\Rightarrow \iint_D f(x,y)\,dA = \iint_S f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta$$

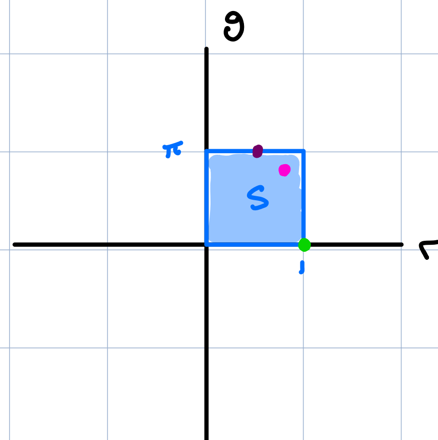
where S = region in $r\theta$-plane that corresponds to D in xy-plane



$x(r,\theta) = r\cos\theta$

$y(r,\theta) = r\sin\theta$

$\longleftarrow$



Defn: A transformation from the uv-plane to the xy-plane is a map

$$T(u,v) = (x(u,v), y(u,v)) = (x,y)$$

↳ Assign point in uv-plane to point in xy-plane.

↳ $x(u,v)$ determines x-coord.

↳ $y(u,v)$ " y-coord.

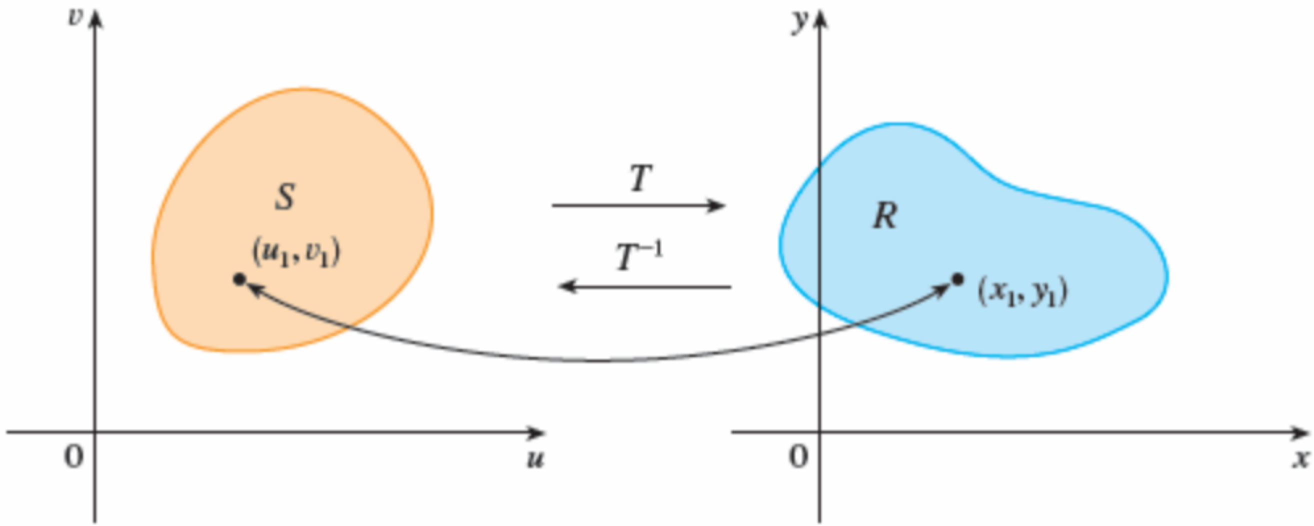
↳ T = fcn w/ domain = $\mathbb{R}^2$ and range = $\mathbb{R}^2$

Ex:

1) $T(r,\theta) = (r\cos(\theta), r\sin(\theta)) = (x,y)$

2) $T(u,v) = (u+v, u-v) = (x,y)$

3) $T(u,v) = (e^{7u}, e^{uv}\cdot\sin(u^2)) = (x,y)$


v
S
(u_1, v_1)
0
u
T
T^{-1}
y
R
(x_1, y_1)
0
x

Defn:

1) If $T(u_1, v_1) = (x_1, y_1)$, then (x_1, y_1) is said to be the image of the point (u_1, v_1).

2) If no two points have the same image, then T is said to be one-to-one.

3) If T transforms S in uv-plane into R in xy-plane, then R is called the image of S under T.

Ex:

1) $T(r, \theta) = (r\cos(\theta), r\sin(\theta))$

i) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is the image of $(1, \pi/4)$

ii) T is <u>not</u> one-to-one: $T(0,0) = T(0,\pi)$

2) $T(u,v) = (u+v, u-v)$

i) Image of $(1,1)$ is $(2,0)$

ii) T is one-to-one: If same image $\Rightarrow$

$$(u_1+v_1, u_1-v_1) = (u_2+v_2, u_2-v_2)$$

$\Rightarrow u_1 = v_1 + u_2 - v_2 = v_1 + u_1 + v_1 - v_2 - v_2$

$\Rightarrow v_1 = v_2$

$\Rightarrow u_1 = u_2$

$\Rightarrow$ actual the same point.

Remark: If T is one-to-one, then it has an inverse

T^{-1}: xy-plane $\to$ uv-plane. So $u(x,y) = u$, $v(x,y) = v$

Ex: Consider $T(u,v) = (u^2 - v^2, 2uv) = (x,y)$

What is the image of $S = [0,1] \times [0,1]$ under T?

Soln: ⓪ Determine where boundary curves go.

$\hookrightarrow$ a) $(u,0) \rightsquigarrow (u^2, 0) = (x,y)$

b) $(u,1) \rightsquigarrow (u^2-1, 2u) = (x,y)$

c) $(0,v) \rightsquigarrow (-v^2, 0) = (x,y)$

d) $(1,v) \rightsquigarrow (1-v^2, 2v) = (x,y)$

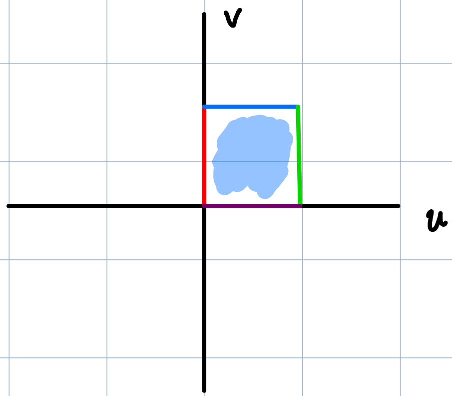
① Solve these parametric equations to get images

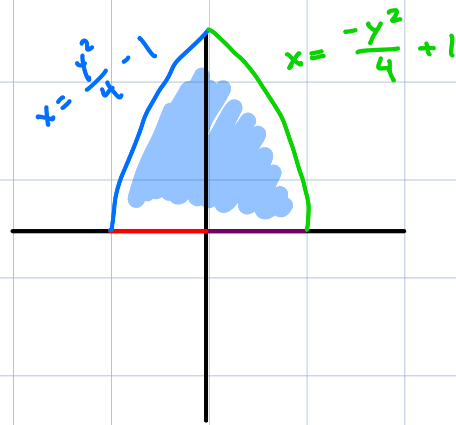
a) $x = u^2, \ y = 0 \implies \{ 0 \le x \le 1, \ y = 0 \}$

b) $x = u^2 - 1, \ y = 2u \implies x = \frac{y^2}{4} - 1$

etc.

② Sketch curves





③ Describe region: $\left\{ \frac{y^2}{4} - 1 \le x \le \frac{-y^2}{4} + 1 ,\ 0 \le y \le 2 \right\}$

Defn: The derivative of a transformation

$$T(u,v) = (x(u,v), y(u,v))$$

is the matrix

$$DT = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

The Jacobian of T is the determinant of DT

$$J(T) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

Theorem: (Change of variables)

Let T be a (C^1) transformation w/ non-zero Jacobian.

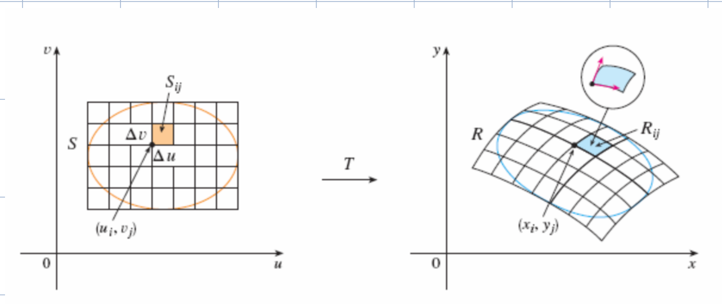
Spse T maps S in uv-plane to R is xy-plane.

Spse T is one-to-one away from the boundary of S.

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \cdot |J(T)| \cdot du\,dv$$

Idea: Show $\Delta A = \underline{|J(T)|} \cdot \Delta u \Delta v$

↪ stretch factor.



Example:

$$T(r,\theta) = (r\cos(\theta),\ r\sin(\theta))$$

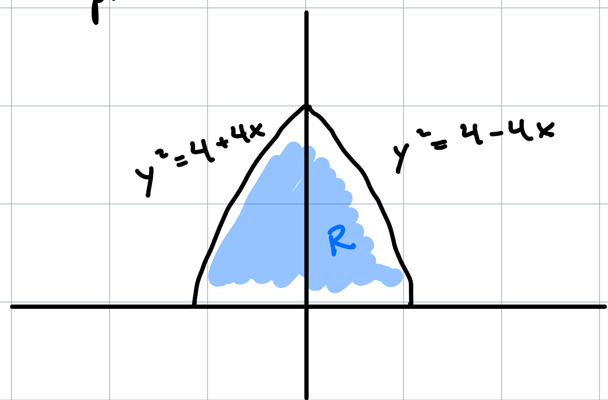
$$DT = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & +r\cos(\theta) \end{pmatrix}$$

$$J(T) = r\cos^2\theta + r\sin^2\theta = r$$

$$\Rightarrow \iint_R f(x,y)\, dA = \iint_S r \cdot f(r\cos(\theta), r\sin(\theta))\, dr\, d\theta$$

Ex: Consider $x = u^2 - v^2$, $y = 2uv$ and evaluate $\iint_R y \, dA$ where R is bounded the x-axis, $y^2 = 4-4x$, and $y^2 = 4+4x$, $y \geq 0$.

Soln: ⓪ Draw picture



① T maps $[0,1] \times [0,1]$ onto R from before

$$\Rightarrow \iint_R y \, dA = \int_0^1 \int_0^1 2uv \cdot |J(T)| \, du dv$$

(2) Compute DT and $J(T)$

$$DT = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$$

$$\Rightarrow \quad J(T) = 4u^2 + 4v^2$$

(3) Plug in and solve

$$\iint_R y \, dA = \int_0^1 \int_0^1 8u^3v + 8v^3u \; du\,dv$$

$$= 8 \int_0^1 \left(\frac{u^4 v}{4} + \frac{u^2 v^3}{2} \right) \Big|_0^1 \, dv$$

$$= 8 \int_0^1 \frac{v}{4} + \frac{v^3}{2} \, dv$$

$$= 8 \cdot \left(\frac{v^2}{8} + \frac{v^4}{8} \right) \Big|_0^1$$

$$= 2$$

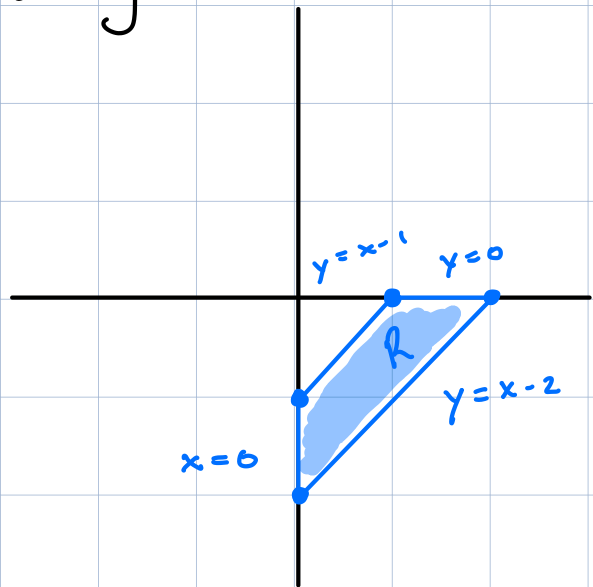
Remark: Pick transformation that will either

i) Simplify integrand, or

ii) Simplify region of integration

Example: Evaluate $\iint_R \exp\left((x+y)/(x-y)\right)\, dA$ where R is trapizoid w/ vertices $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$.

Soln: ⓪ Draw region



① Pick transformation: $u = x+y$, $v = x-y$

$\Rightarrow T(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) = (x,y)$

$T^{-1}(x,y) = (x+y, x-y) = (u,v)$

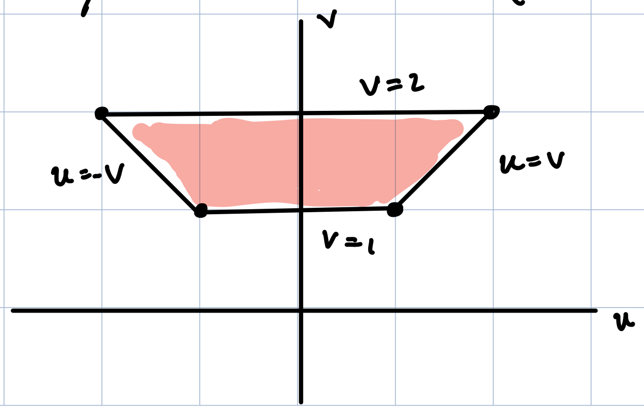
② Compute new region: Image of R under T

$x=0 \Rightarrow u=-v \quad (1 \le y \le 2 \Rightarrow 1 \le u \le 2)$

$y=0 \Rightarrow u=v \quad (1 \le x \le 2 \Rightarrow 1 \le u \le 2)$

$x-y=2 \Rightarrow v=2 \quad (0 \le x \le 1 \Rightarrow -2 \le u \le 2)$

$x-y=1 \Rightarrow v=1 \quad (0 \le x \le 2 \Rightarrow -1 \le u \le 2)$



③ Compute Jacobian of T

$$J(T) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = -\frac{1}{2}$$

④ Set up integral and solve

$$\iint_R f \, dA = \int_1^2 \int_{-v}^{v} \frac{1}{2} \exp(u/v) \, du\, dv$$

$$= \int_1^2 \frac{1}{2} \cdot v \exp(u/v) \Big|_{-v}^{v} \, dv$$

$$= \int_1^2 \frac{1}{2} v \exp(1) - \frac{1}{2} v \exp(-1) \, dv$$

$$= \frac{1}{2}(e - e^{-1}) \cdot \frac{v^2}{2} \Big|_1^2$$

$$= \frac{1}{2}(e - e^{-1}) \cdot \frac{3}{2}$$