Lecture # 7

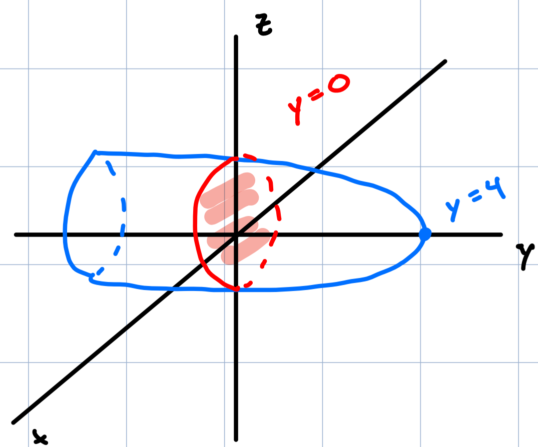
Title: Triple integrals in cylindrical and spherical coordinates

Section: Stewart 15.7, 15.8

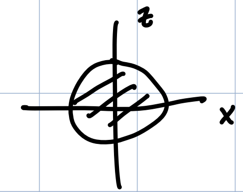
Warm-up: Spse E is bounded by $y = 4 - x^2 - z^2$ and $y = 0$

Express the volume of E as 6 different iterated integrals.

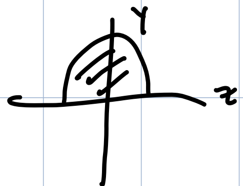
Soln: ⓪ Draw picture



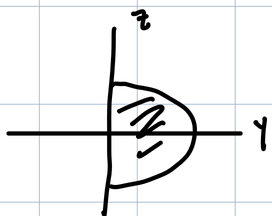
① Set up integrals:



- $$\int_{-2}^{2}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{0}^{4-x^2-z^2} dy\,dx\,dz$$



- $$\int_{-2}^{2}\int_{0}^{4-z^2}\int_{-\sqrt{4-y-z^2}}^{\sqrt{4-y-z^2}} dx\,dy\,dz$$



- $$\int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-y-z^2}}^{\sqrt{4-y-z^2}} dx\,dz\,dy$$

- The others are similar.

Fact: $\iiint_E 1\, dV =$ volume of E.

Cylindrical Coordinates

Defn: Cylindrical coordinate is a triple (r, θ, z) where

↪ (r, θ) are polar dir. in xy-plane

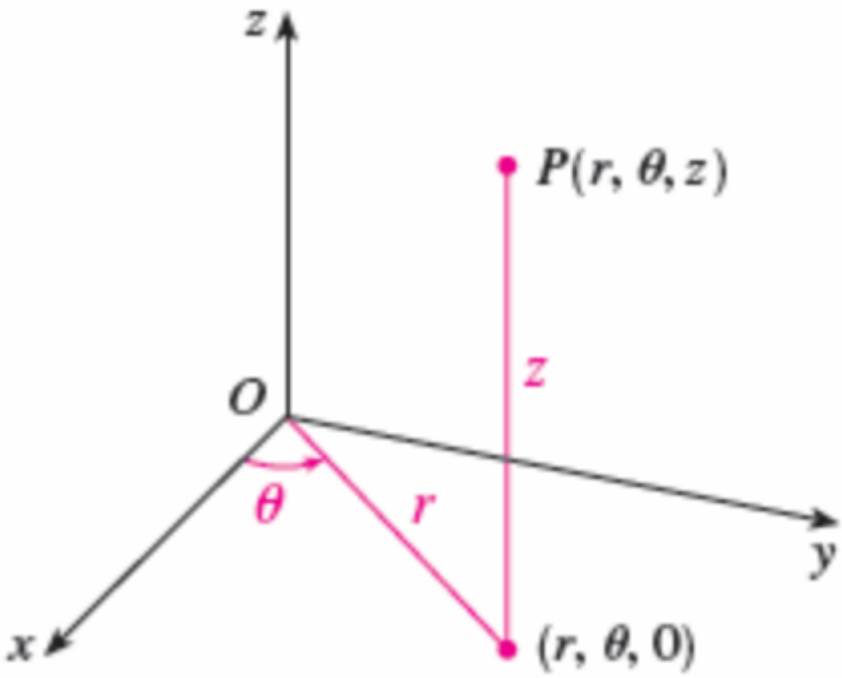
z = height of point above xy-plane

Fact: Polar to Rectangular

- $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$

Rect to Polar

- $r^2 = x^2 + y^2$, $\tan(\theta) = y/x$, $z = z$



z
$P(r, \theta, z)$
z
O
θ
r
y
x
$(r, \theta, 0)$

Ex:

1) Describe $r = \text{constant}$ = cylinder

2) " $z = r$ = cones

3) $r^2 + z^2 = 4$ = sphere

Triple integrals w/ polar coordinates

Theorem: Spse $E = \{(x,y,z) \mid (x,y) \text{ in } D \text{ and } u_1(x,y) \leq z \leq u_2(x,y)\}$
w/ $D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta \text{ and } h_1(\theta) \leq r \leq h_2(\theta)\}$,

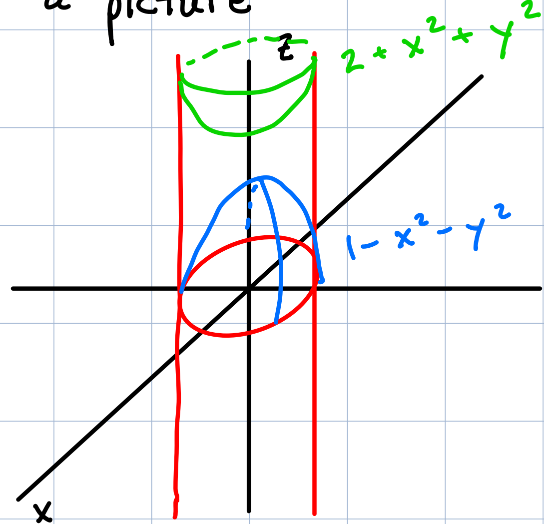
$$\iiint_E f(x,y,z)\,dV$$
$$= \iint_D \left(\int_{u_1}^{u_2} f(x,y,z)\,dz\right) dA$$
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} r \cdot f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Montra: replace x w/ $r\cos(\theta)$, y w/ $r\sin(\theta)$, leave z,
dV w/ $r\,dz\,dr\,d\theta$

Idea: Do polar rectangle type setup (15.3) w/ 3-var. integrals.

Example: Compute $\iiint_E \sqrt{x^2+y^2}\, dV$ where E lies inside $x^2+y^2=1$, below $z=2+x^2+y^2$, and above $z=1-x^2+y^2$.

Soln: ⓪ Draw a picture



① Determine bounds

↳ $1 - x^2 - y^2 \leq z \leq 2 + x^2 + y^2$

(x, y) in $D = \{ x^2 + y^2 \leq 1 \}$

② Set up cylindrical integral and solve

$$\iiint_E \sqrt{x^2+y^2}\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^{2+r^2} r^2 \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^2 (2 + r^2 - 1 + r^2) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^2 (1 + 2r^2) \, dr \, d\theta$$

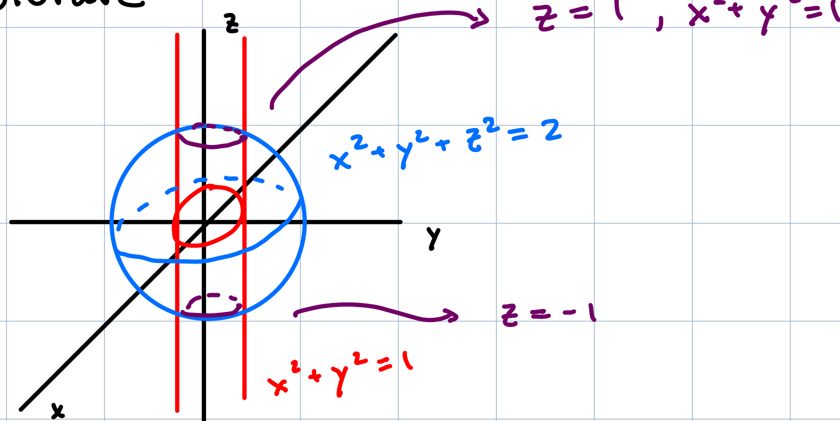
$$= \int_0^{2\pi} \int_0^1 r^2 + 2r^4 \, dr \, d\theta$$

$$= 2\pi \left(\frac{r^3}{3} + \frac{2r^5}{5} \right) \Big|_0^1$$

$$= 2\pi \left(\frac{1}{3} + \frac{2}{5} \right)$$

Example: Find the volume enclosed by $x^2+y^2=1$ and $x^2+y^2+z^2=2$.

Soln: ⓪ Draw picture



① Solve for bounds

$$\hookrightarrow \quad -\sqrt{2-x^2-y^2} \le z \le \sqrt{2-x^2-y^2}$$

$$(x,y) \text{ in } D = \{ x^2+y^2 \le 1 \}$$

② Set up cylindrical integral and solve

$$\text{Volume} = \iiint_E 1 \, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

$$= 2 \int_0^{2\pi} \int_0^1 r \cdot \sqrt{2-r^2} \, dr \, d\theta$$

$$= 4\pi \cdot \int_0^1 r \cdot \sqrt{2-r^2} \, dr$$

$2 - r^2 = u$

$-2r\,dr = du$

$$= -2\pi \int_2^1 \sqrt{u} \, du$$

$$= -2\pi \left(\frac{2}{3} u^{3/2} \right) \Big|_2^1$$

$$= -\frac{4\pi}{3} \left(1 - \sqrt{8} \right)$$

Example: Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$ using cylindrical coordinates.

Soln:

⓪ Identify E

$\hookrightarrow$ E = region between $z=2$ and $z=\sqrt{x^2+y^2}$ over $D=\{x^2+y^2 \le 4\}$

① Set up cylindrical integral and solve

$$\iiint_E f\, dV = \int_0^{2\pi}\int_0^2 \left(\int_r^2 r^3\, dz\right) dr\, d\theta$$

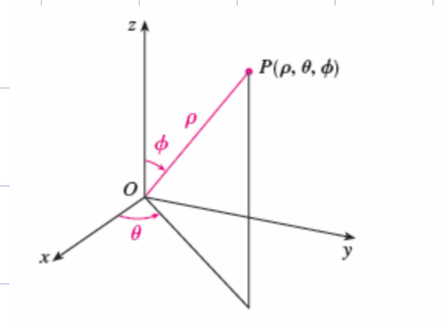
$$= 2\pi \int_0^2 2r^3 - r^4 \quad dr\, d\theta$$

$$= 2\pi \left(\frac{r^4}{2} - \frac{r^5}{5}\right)\Big|_0^2$$

$$= 2\pi\left(8 - \frac{32}{5}\right)$$

Spherical coordinates

Picture:



Notn:

ρ = distance from the origin $(0 \le \rho)$

θ = angle wrt x-axis $(0 \le \theta \le 2\pi)$

ϕ = angle downward from positive z-axis.

Example: • $\{\rho = c\}$ = sphere of radius c.

• $\{\theta = c\}$ = vertical half-plane

• $\{\phi = c\}$ = cone centered along z-axis

Figure 2

$\rho = c$, a sphere

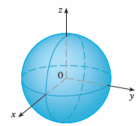


Figure 3

$\theta = c$, a half-plane

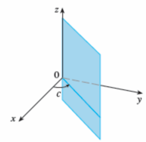
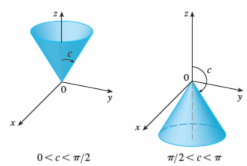


Figure 4

$\phi = c$, a half-cone



Fact: Spherical to polar

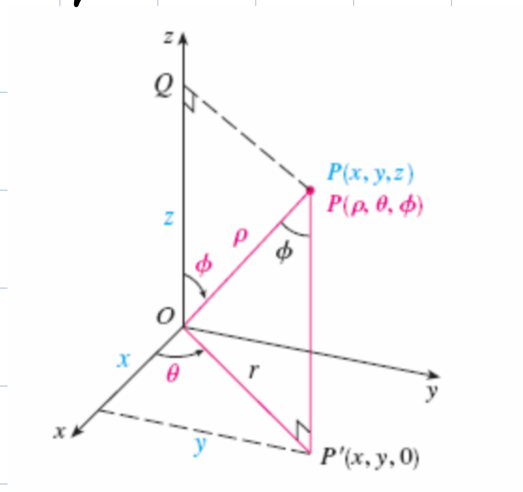
- $x = \rho \sin(\phi) \cos(\theta)$

  $y = \rho \sin(\phi) \sin(\theta)$

  $z = \rho \cos(\theta)$

  $\rho^2 = x^2 + y^2 + z^2$

Picture:



"Proof": SOHCAHTOA

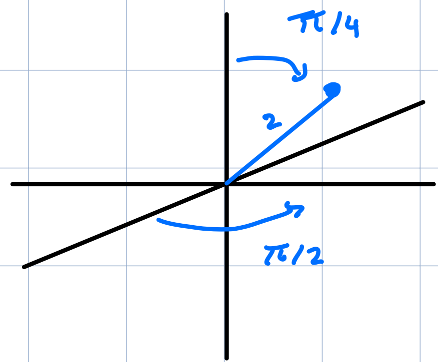
$\hookrightarrow$ Ex: $x = r\cos(\theta) = \rho \sin(\phi) \cos(\theta)$ $\square$

Example: 1) What is the Cartesian coord for $(2, \pi/2, \pi/4)$

$$\hookrightarrow \quad x = 2 \cdot \sin(\pi/4) \cdot \cos(\pi/2) = 0$$

$$y = 2 \cdot \sin(\pi/4) \cdot \sin(\pi/2) = \sqrt{2}$$

$$z = 2 \cdot \cos(\pi/4) = \sqrt{2}$$



Ex: 2) What is the spherical coord. for $(0, 2\sqrt{3}, -2)$?

$\hookrightarrow \rho = \sqrt{0 + (2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4$

$\cos(\phi) = z/\rho = -1/2 \Rightarrow \phi = \frac{2\pi}{3}$ * $0 \le \phi \le \pi$

$\cos(\theta) = x/(\rho \sin(\phi)) = 0 \Rightarrow \theta = \pi/2$ *

* y-coord is pos $\Rightarrow \theta \neq \frac{3\pi}{2}$

Fact: If $E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$, then

$$\iiint_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b \rho^2 \sin(\phi) \cdot f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)\, d\rho\, d\theta\, d\phi.$$

$\hookrightarrow$ $x \rightsquigarrow \rho \sin\phi \cos\theta$

$y \rightsquigarrow \rho \sin\phi \sin\theta$

$z \rightsquigarrow \rho \cos\phi$

$dV \rightsquigarrow \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi.$

Ex: Evaluate $\iiint_B \exp\left((x^2+y^2+z^2)^{3/2} \right) dV$ where
$B = $ unit ball $= \{ (x,y,z) \mid x^2+y^2+z^2 \le 1 \}$.

Soln: ① Find spherical bounds for B.

$\rightsquigarrow \quad B = \{ (\rho, \theta, \phi) \mid 0 \le \rho \le 1 \}$

② Use spherical integral formula

$$\text{integral} = \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \sin(\phi) \exp(\rho^3) \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin(\phi) \cdot \left(\exp(\rho^3)/3 \right) \Big|_0^1 \, d\theta \, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin(\phi) \cdot \tfrac{1}{3}(e-1) \, d\theta \, d\phi$$

$$= \int_0^{\pi} (2\pi/3)(e-1) \sin(\phi) \, d\phi$$

$$= -\frac{2\pi}{3}(e-1) \cdot \cos(\phi) \Big|_0^{\pi}$$

$$= \frac{4\pi}{3}(e-1)$$

Fact: If $E = \{ \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi) \}$

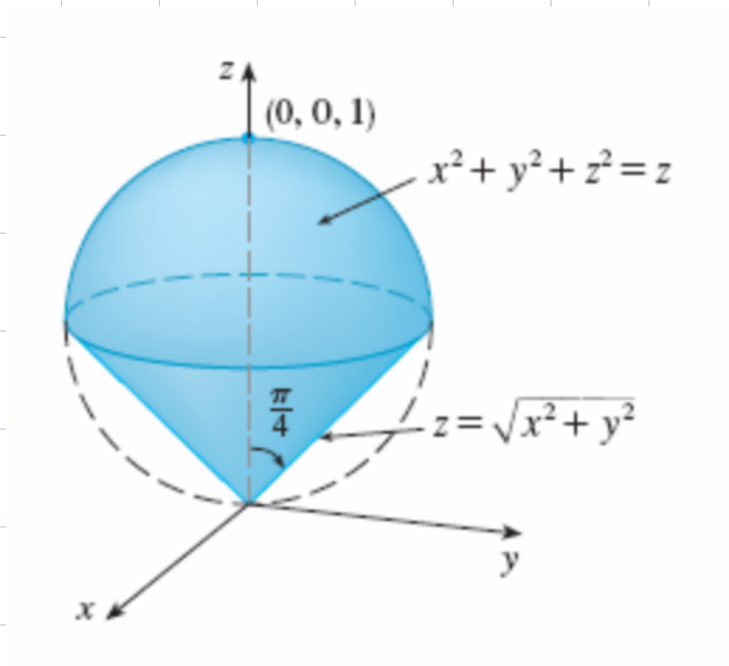
then

$$\iiint_E f(x,y,z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} \rho^2 \sin(\phi) \cdot f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho\cos\phi)\, d\rho\, d\theta\, d\phi$$

Example: Find volume of solid above $z = \sqrt{x^2+y^2}$ and below $z^2 + x^2 + y^2 = z$

Soln: ⓪ Draw picture



① Convert equation to spherical coords

- $\rho^2 = \rho\cos(\phi) \Rightarrow \rho = \cos(\phi)$
- $\rho\cos(\phi) = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\cos^2\theta}$

$$= \rho\sin(\phi)$$

$$\Longleftrightarrow \cos(\phi) = \sin(\phi) \Longleftrightarrow \phi = \pi/4$$

② Find bounds:

- $0 \le \rho \le \cos(\phi)$
- $0 \le \theta \le 2\pi$
- $0 \le \phi \le \pi/4$

(7) Set up integral and solve

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$2\pi \int_0^{\pi/4} \left(\frac{\rho^3}{3} \sin(\phi)\right)\Big|_0^{\cos(\phi)} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \cos^3(\phi) \sin(\phi)\, d\phi$$

$u = \cos(\phi)$

$du = -\sin(\phi)\, d\phi$

$$= -\frac{2\pi}{3} \int_1^{\sqrt{2}/2} u^3\, du$$

$$= \frac{-2\pi}{12} \cdot u^4 \Big|_1^{\sqrt{2}/2}$$

$$= \frac{-\pi}{6}\left(\frac{1}{4} - \frac{4}{4}\right)$$

$$= \frac{\pi}{8}$$