Lecture # 6

Title: Triple integrals

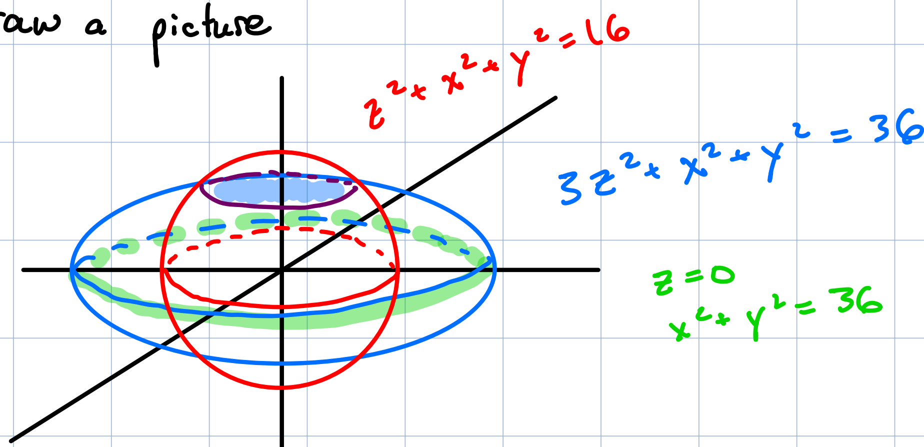
Section: Stewart 15.6

Warm-up: Find an integral expression for the surface area of $3z^2 + x^2 + y^2 = 36$ inside of $z^2 + x^2 + y^2 = 16$.

Use polar coordinates. (You can actual integrate...)

Soln:

① Draw a picture



② Where do these intersect?

$$3z^2 - 36 = z^2 - 16 \Rightarrow z^2 = 10$$

$$\Rightarrow x^2 + y^2 = 6 \Rightarrow r^2 = 6$$

③ Phrase as integral. *For top + bottom.

$$SA = {}^*2 \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

where $D = \{ r \le \sqrt{6} \}$

$$f(x,y) = \sqrt{(36 - x^2 - y^2)/3}$$

④ Compute f_x, f_y

$$f_x = \frac{1}{\sqrt{3}} \cdot \frac{-x}{\sqrt{36 - x^2 - y^2}}$$

$$f_y = \frac{1}{\sqrt{3}} \cdot \frac{-y}{\sqrt{36 - x^2 - y^2}}$$

⑤ $$SA = \int_0^{\pi} \int_0^{\sqrt{6}} r \cdot \sqrt{1 + \frac{r^2 \cos^2(\theta)}{36 - r^2} + \frac{r^2 \sin^2\theta}{36 - r^2}}\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{\sqrt{6}} r \cdot \sqrt{36/(36 - r^2)}\, dr\, d\theta$$

Recall:

- $\int f(x)\,dx$ = "signed" area under graph of f
- $\iint f(x,y)\,dA$ = " volume " " " "
- $\iiint f(x,y,z)\,dV$ = " "4D volume" " " " "

  ↪ Graph of $f(x,y,z)$ lives in $\mathbb{R}^4$.

  ↪ Want to defined this!

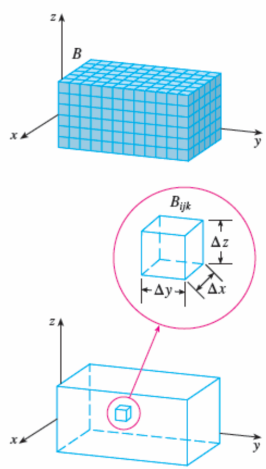
Notn:

- $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$
- $B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ s \le z \le t\}$

Setup:

- Divide $[a,b]$ into subintervals $[x_{i-1}, x_i]$ of width Δx
- " $[c,d]$ " " $[y_{j-1}, y_j]$ " " Δy
- " $[s,t]$ " " $[z_{k-1}, z_k]$ " " Δz.
- $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$
- Sample (x_i^*, y_j^*, z_k^*) in B_{ijk}
- $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$

Picture:



Definition: $\iiint_B f(x,y,z)\,dV$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*) \cdot \Delta V$$

Theorem: (Fubini) $\iiint_B f(x,y,z)\,dV$

$$= \int_s^t \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

The ordering of doesn't change the integral.

↳ B must be a <u>box</u>, not a more general region.

Example: $\iiint_B xyz^2 \, dV$ w/ $B = [0,1] \times [-1,2] \times [0,3]$

Soln: ① Write as iterated integrals and solve

$$\iiint_B xyz^2 \, dV = \int_0^1 \int_{-1}^2 \int_0^3 xyz^2 \, dz \, dy \, dx$$

$$= \int_0^1 \int_{-1}^2 \left(xyz^3/3 \right) \Big|_0^3 \, dy \, dx$$

$$= \int_0^1 \int_{-1}^2 9xy \, dy \, dx$$

$$= \int_0^1 \left(9xy^2/2 \right) \Big|_{-1}^2 \, dx$$

$$= \int_0^1 18x - \frac{9}{2}x \, dx$$

$$= \left(9x^2 - (9/4)x^2 \right) \Big|_0^1$$

$$= 9 - 9/4$$

$$= \frac{36}{4} - \frac{9}{4}$$

$$= 27/4$$

Notn:

- Let E = general bounded region.
- Let B = box containing E
- $$F(x,y,z) = \begin{cases} f(x,y,z) & , (x,y,z) \in E \\ 0 & , \text{else} \end{cases}$$

Defn: $$\iiint_E f \, dV = \iiint_B F \, dV.$$

↳ Recall this setup from 15.2!

↳ In general, computing LHS can be extremely difficult.

So as in 15.2, we will consider nice types of regions.

Defn: E is of type 1 if

$$E = \{ (x,y) \text{ in } D \text{ and } u_1(x,y) \leq z \leq u_2(x,y) \}$$

$\hookrightarrow$ So solid E lies over some $D \subseteq \mathbb{R}^2$ and between two graphs of two fcns in x, y.

Example:
- $E = \{ 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x^2 + y^2 \leq z \leq 5x^2 + 5y^2 \}$
- * $E = \{ a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ u_1(x,y) \leq z \leq u_2(x,y) \}$
- $E = \{ x^2 + y^2 \leq 5,\ 2x - \sin(y) \leq z \leq 27e^{xy} \}$

Theorem: If E is type I, then

$$\iiint_E f\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f \, dz \right) dA$$

Cor: If E is * above, then

$$\iiint_E f\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f \, dz\, dy\, dx$$

$\hookrightarrow$ So D is type I from 15.2

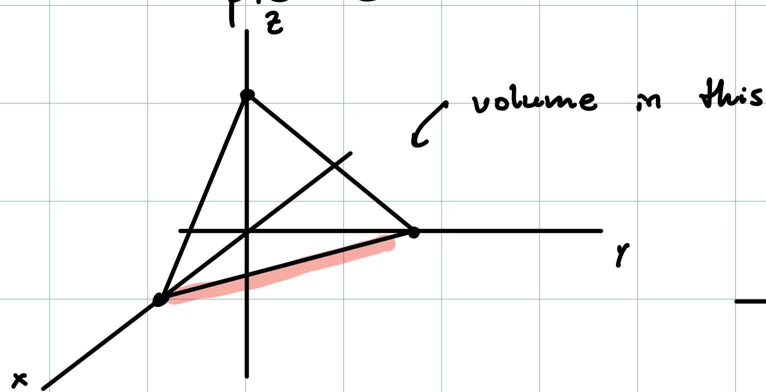
Analogous formula if D is type II.

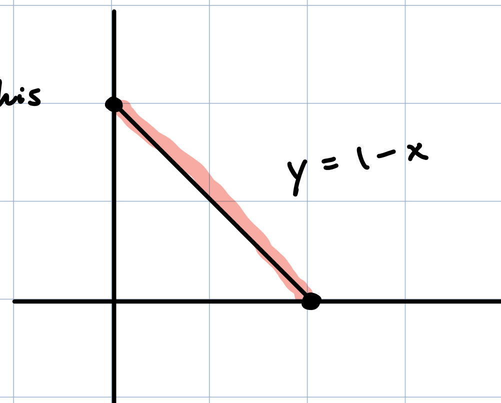
Ex: Compute $\iiint_E z \, dV$ where E is solid bounded by

$x=0$, $y=0$, $z=0$, and $x+y+z=1$

Soln:

⓪ Draw picture





① Establish bounds

$\hookrightarrow 0 \leq z \leq 1-x-y$

$\hookrightarrow 0 \leq y \leq 1-x$

$\hookrightarrow 0 \leq x \leq 1$

② Set up integral and solve

$$\iiint_E z \, dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \; dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{z^2}{2}\right)\Big|_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2 \; dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{-1}{3}(1-x-y)^3\right)\Big|_0^{1-x} dx$$

$$= \frac{-1}{6}\int_0^1 (1\cancel{-x-1+x})^3 - (1-x)^3 \, dx$$

$$= \frac{1}{6}\left(\frac{1}{4}(1-x)^4\right)\Big|_0^1$$

$$= \frac{1}{24}$$

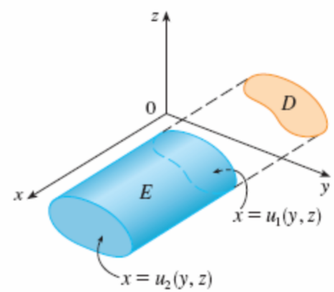
Fact: Spse $E = \{(y,z) \text{ in } D \subseteq yz\text{-plane},\ u_1(y,z) \leq x \leq u_2(y,z)\}$

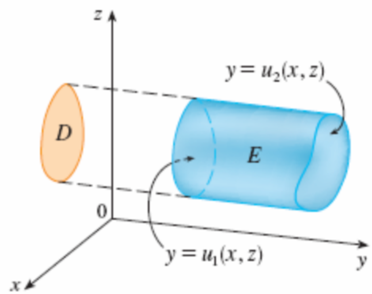
then $$\iiint_E f\, dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f\, dx \right) dA$$

↳ think of x as "up direction".

↳ See type III in book for y as "up direction"

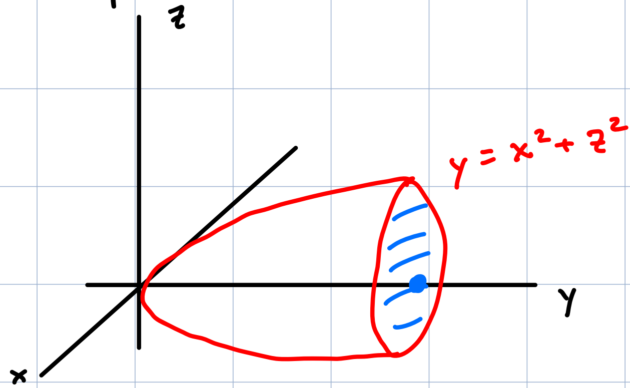
Picture:

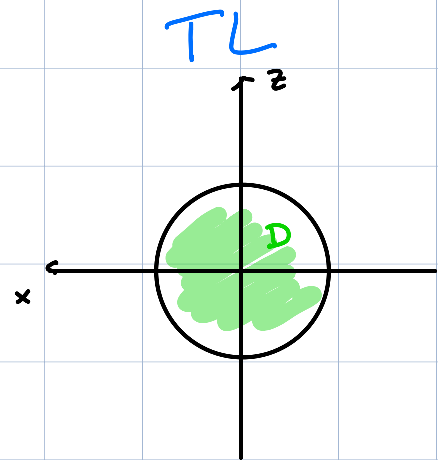




Example: Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$ where E is bounded by $y = x^2 + z^2$ and $y = 4$

Soln: ① Draw picture





② Find bounds

↪ $x^2 + z^2 \le y \le 4$

↪ (z, x) in $D = \{ x^2 + z^2 \le 4 \}$

③ Set up integral and solve

$$\iiint_E \sqrt{x^2+z^2}\, dV$$

$$= \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dV \right)$$

$$= \iint_D \left(y \cdot \sqrt{x^2+z^2} \right)\Big|_{x^2+z^2}^{4} dA$$

$$= \iint_D 4\sqrt{x^2+z^2} - (x^2+z^2)^{3/2}\, dA$$

④ Use polar coordinates for x and z.

$\hookrightarrow D = \{ r \leq 2 \}$, $x = r\cos\theta$, $z = r\sin(\theta)$

$$\text{integral} = \int_0^{2\pi} \int_0^2 r(4r - r^3)\, dr\, d\theta$$

$$= 2\pi \left(\frac{4}{3} r^3 - \frac{r^5}{5} \right)\Big|_0^2$$

$$= 2\pi \left(\frac{32}{3} - \frac{32}{5} \right)$$

$$= 128\pi/15$$

Soln 2:

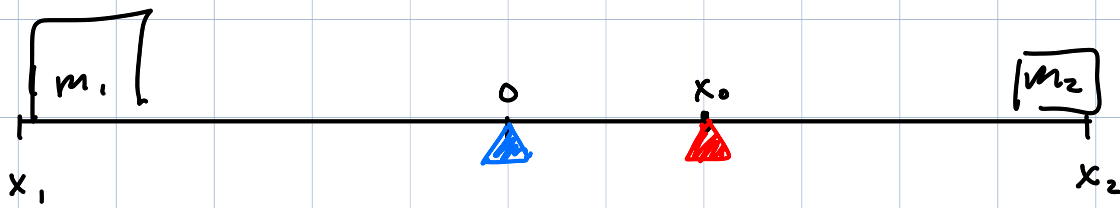
$$E = \{ -\sqrt{y-x^2} \le z \le \sqrt{y-x^2},$$
$$-\sqrt{y} \le x \le \sqrt{y}$$
$$0 \le y \le 4 \}$$

$$\Rightarrow \iiint_E \sqrt{x^2+z^2}\, dV = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dx\, dy$$

↳ So some orders are nicer than others

Physical interpretation of $\iiint_E f \, dV$

Motivation:

- 

- Moment $= x_1 m_1 + x_2 \cdot m_2$

  ↳ If pos. $\Rightarrow$ will swing clockwise on

  ↳ If neg $\Rightarrow$ " " counter-clockwise on

- When $m_1 \cdot (x_1 - x_0) + m_2 \cdot (x_2 - x_0) = 0$, then system will be balanced on .

$0 = \quad = \text{moment} - m_1 \cdot x_0 - m_2 \cdot x_0 = \text{moment} - (m_1 + m_2) x_0$

$\Rightarrow x_0 = \text{moment} / (\text{total mass})$

Such an x_0 is the center of mass.

Notn:

- E = region in $\mathbb{R}^3$
- $\rho(x,y,z)$ = mass per unit volume (density)

Defn:

- $\iiint_E \rho \, dV$ = mass of E wrt density ρ.
- The moments of E wrt ρ are

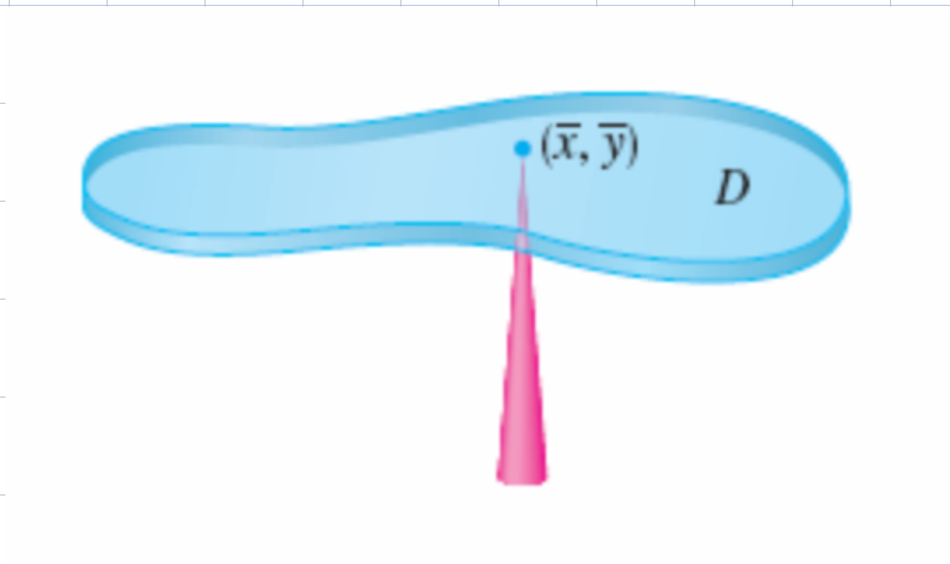
$$\left.\begin{aligned} M_{yz} &= \iiint_E x \cdot \rho \, dV \\ M_{xz} &= \iiint_E y \cdot \rho \, dV \\ M_{xy} &= \iiint_E z \cdot \rho \, dV \end{aligned}\right\} \text{"Some measure of tendency for } E \text{ to rotate about origin in } \mathbb{R}^4\text{"}$$

The center of mass is the point

$$\left(\frac{M_{yz}}{\iiint_E \rho \, dV}, \frac{M_{xz}}{\iiint_E \rho \, dV}, \frac{M_{xy}}{\iiint_E \rho \, dV} \right)$$

$\hookrightarrow$ balance point

Picture:



Example: Spse E is the cube $[0,1]\times[0,1]\times[0,1]$ w/ density $\rho(x,y,z) = x^2+y^2+z^2$.

What is the center of mass?

Soln: ① Find mass:

$$\text{Mass} = \int_0^1\int_0^1\int_0^1 x^2+y^2+z^2 \, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 \left(\frac{x^3}{3} + xy^2 + xz^2\right)\Big|_0^1 \, dy\, dz$$

$$= \int_0^1\int_0^1 \left(\frac{1}{3} + y^2 + z^2\right) dy\, dz$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^3}{3} + yz^2\right)\Big|_0^1 \, dz$$

$$= \int_0^1 \frac{2}{3} + z^2 \, dz$$

etc.

$$= 1$$

② Find moments.

↪ By symmetry $M_{xy} = M_{xz} = M_{yz}$.

↪ $$M_{xy} = \int_0^1 \int_0^1 \int_0^1 x^3 + xy^2 + xz^2 \, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \left(\frac{x^4}{4} + \frac{x^2 y^2}{2} + \frac{x^2 z^2}{2} \right)\Big|_0^1 \, dy\, dz$$

$$= \int_0^1 \int_0^1 \frac{1}{4} + \frac{y^2}{2} + \frac{z^2}{2} \, dy\, dz$$

$$= \frac{1}{4} + \frac{1}{3}$$

$$= 7/12$$

⇒ center of mass $= (7/12, 7/12, 7/12)$.