Lecture # 5

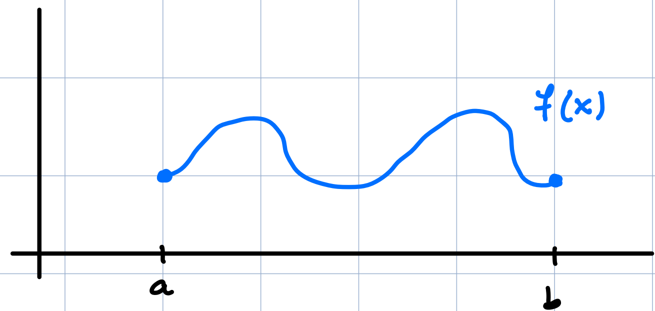
Title: Surface Area

Section: Stewart 15.5

Lengths of curves

Notn:

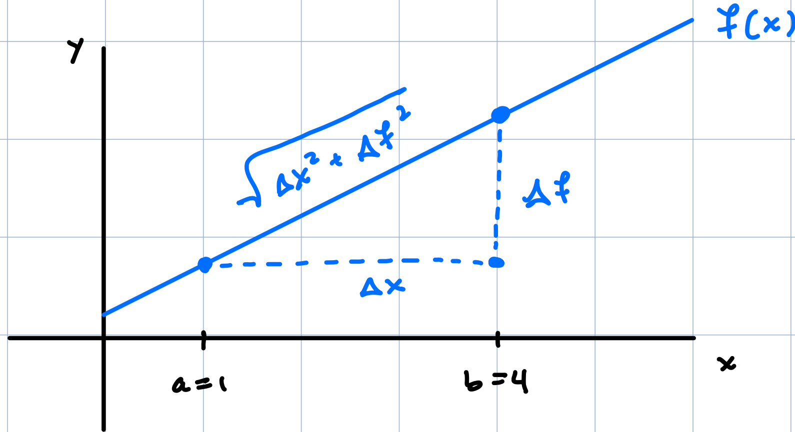
- $f : [a, b] \longrightarrow \mathbb{R}$ = differentiable fcn
- The graph of f is a curve in $\mathbb{R}^2$



Question: What is the length of this "curve/graph"

Answer: $\text{Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

Rem: If $f(x) = m \cdot x + b$, then



$$\text{Length from } a \text{ to } b = \sqrt{\Delta x^2 + \Delta f^2}$$

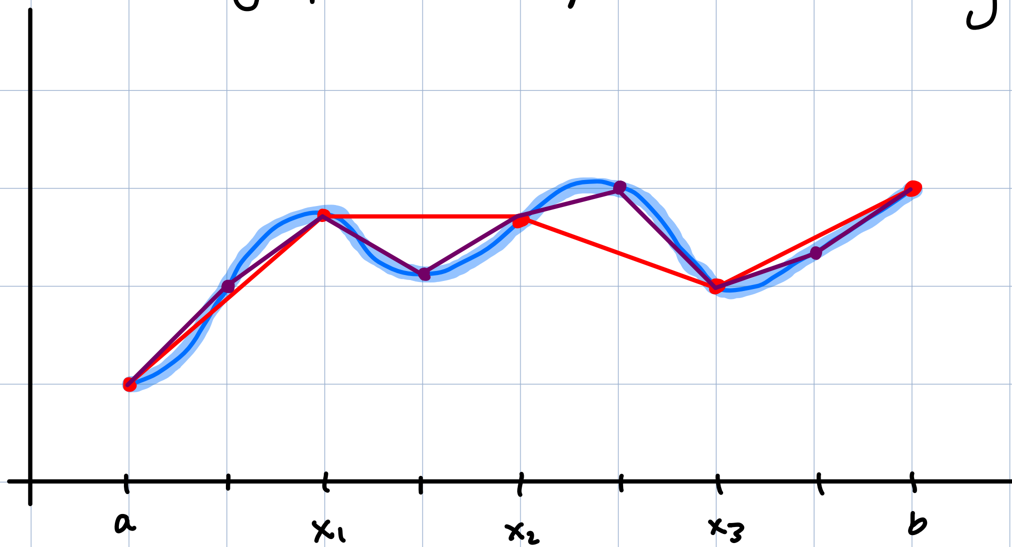
$$= \Delta x \cdot \sqrt{1 + \Delta f^2 / \Delta x^2}$$

$$= (b-a) \cdot \sqrt{1 + m^2}$$

$$= \int_a^b \sqrt{1 + (f'(x))^2}$$

Divide $[a, b]$ up into n subintervals $[x_{i-1}, x_i]$ w/ widths Δx_i. Set $\Delta f_i = f(x_i) - f(x_{i-1})$

Approximate the graph of f by secant line segments



As $n \to \infty$, red segments hug graph of f more and more

$\Rightarrow$ length of red segments approx. length of graph

Reimann sum : $\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta f_i^2}$ = length of line segments

$$= \sum_{i=1}^{n} \sqrt{1 + (\Delta f_i / \Delta x_i)^2} \; \Delta x_i$$

$$= \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2} \cdot \Delta x_i$$

Some sample x_i^* given by mean value thm

$$= \text{Reimann sum for } \sqrt{1 + (f')^2}$$

$$\Rightarrow \text{Length} = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

Example: What is the length of the graph of $f(x) = \sqrt{1-x^2}$ from -1 to 1?

Soln: ① Compute f': $f'(x) = -x/\sqrt{1-x^2}$

② Use formula:

$$\text{length} = \int_{-1}^{1} \sqrt{1 + x^2/(1-x^2)}\, dx$$

$$= \int_{-1}^{1} \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}}\, dx$$

$$= \int_{-1}^{1} \sqrt{1/(1-x^2)}\, dx$$

$$= \arcsin(x) \Big|_{-1}^{1}$$

$$= \pi/2 - (-\pi/2)$$

$$= \pi$$

Question: Given graph of $f(x,y)$, how do we compute its surface area?

Review of partial derivatives

Notn: $f(x,y)$ = fcn in two variables

Defn: The partial derivative of f wrt x is

$f_x = \partial f/\partial x$ = x-derivative of f w/ viewing y as constant.

The partial derivative of f wrt y is

$f_y = \partial f/\partial y$ = y-derivative of f w/ viewing x as constant.

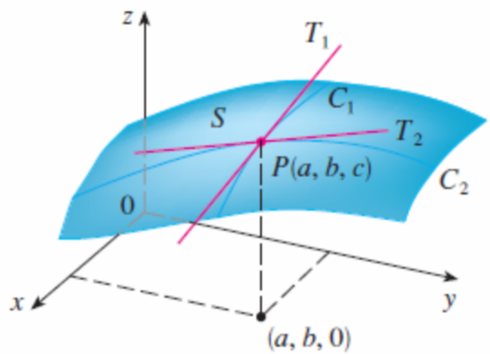
Example: $$f(x,y) = 3x^2y + \sin(xy)$$

$$\partial f/\partial x = 6xy + y\cos(xy)$$

$$\partial f/\partial y = 3x^2 + x\cos(xy)$$

Picture:

The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.



Rem: $\partial f/\partial x$ = infintesimal change of f in x-direction

$\partial f/\partial y$ = " " " f " y- "

Surface Area

Theorem: The surface area of the graph of f over a region D is

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Idea: 1) given a plane $ax + by + c = f(x, y)$, the SA over a rectangle $[x_0, x_1] \times [y_0, y_1]$ is

$$\sqrt{1 + a^2 + b^2} \cdot \Delta x \cdot \Delta y$$

where $\Delta x = x_1 - x_0$, $\Delta y = y_1 - y_0$

↳ Why? : We use that the area of parallelogram spanned by vectors $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.

↳ $\vec{a} = \Delta x \, \vec{i} + a \cdot \Delta x \cdot \vec{k}$, $\vec{b} = \Delta y \, \vec{j} + b \cdot \Delta y \cdot \vec{k}$

2) Divide region D up into rectangles R_{ij} of size $\Delta x \cdot \Delta y$

$$SA \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \text{SA of } \underline{a} \text{ tangent plane of } f \text{ over } R_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} SA(T_{ij})$$

*
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{1 + f_x(x_i^*, y_j^*)^2 + f_y(x_i^*, y_j^*)^2} \cdot \Delta x \cdot \Delta y$$

$$= \text{Reimann sum for } \sqrt{1 + f_x^2 + f_y^2}$$

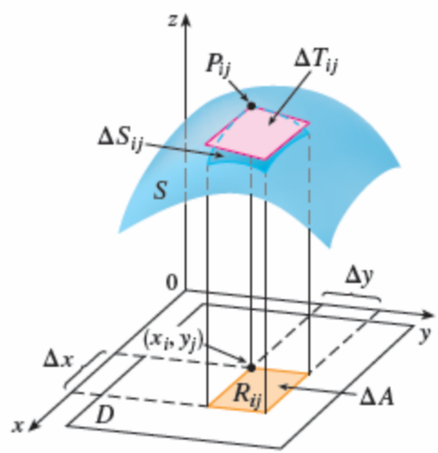
* $T_{ij} = f_x(x_i^*, y_j^*)(x - x_i^*) + f_y(x_i^*, y_j^*)(y - y_j^*) + f(x_i^*, y_j^*)$

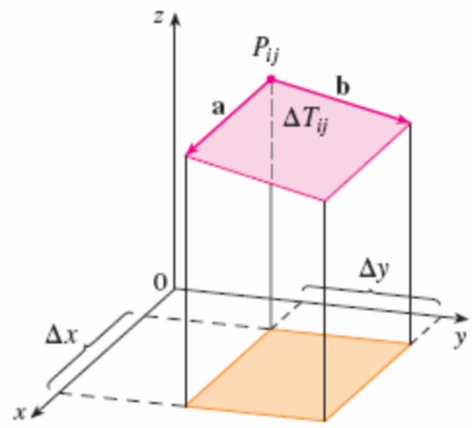
for some samples (x_i^*, y_j^*) in R_{ij}

7) As size of rectangles go to zero Reimann sum becomes

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dA$$

Picture:





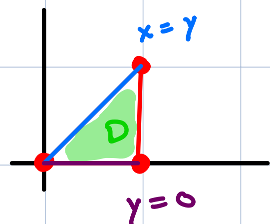
Example: Compute surface area of plane $3x + 2y + z = 6$ that lies above the triangle w/ vertices $(0,0)$, $(1,0)$, $(1,1)$

Soln:

-1 View as integral problem:

SA of graph of $f(x,y) = -3x - 2y + 6$

0 Draw picture of D



1 Determine bounds: $0 \leq y \leq x$, $0 \leq 1 \leq x$

② Set up integral and solve

$$SA = \int_0^1 \int_0^x \sqrt{1 + 9 + 4} \quad dy\, dx$$

$$= \int_0^1 \int_0^x \sqrt{14} \; dy\, dx$$

$$= \sqrt{14} \cdot \int_0^1 y \Big|_0^x \; dx$$

$$= \sqrt{14} \cdot \int_0^1 x \; dy$$

$$= \sqrt{14} \cdot (x^2/2) \Big|_0^1$$

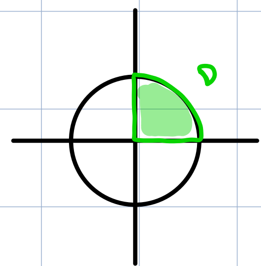
$$= \sqrt{14}/2$$

Exercise:

Compute the surface area of the graph of $f(x,y) = xy$ over the part of the disk of radius 1 in the first quadrant.

Soln:

⓪ Draw picture



① Describible domain in polar coord.

$$D = \{ (r, \theta) \mid 0 \leq r \leq 1, \ 0 \leq \theta \leq \tfrac{\pi}{2} \}$$

② Set up integral and solve

$$SA = \iint_D \sqrt{1 + x^2 + y^2}\, dA$$

$$= \int_0^{\pi/2}\int_0^1 r\sqrt{1 + r^2}\, dr\, d\theta$$

$u = 1 + r^2$

$du = 2r\, dr$

$$= \int_0^{\pi/2}\int_1^2 \frac{1}{2}\sqrt{u}\, du\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} \cdot \frac{2}{3} u^{3/2}\right)\Big|_1^2 d\theta$$

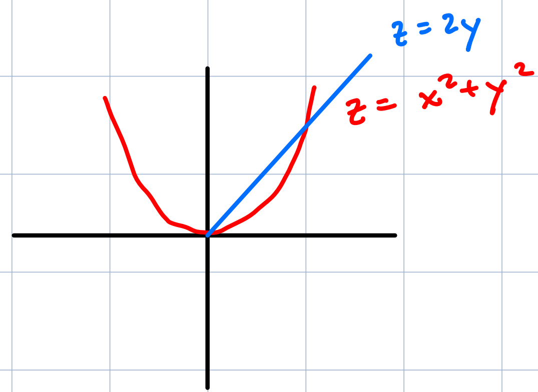
$$= \frac{\pi}{6} \cdot \left(2^{3/2} - 1^{3/2}\right)$$

$$= \frac{\pi}{6} \cdot \left(\sqrt{8} - 1\right)$$

Example: What is the surface area of $z = x^2 + y^2$ that lies under the plane $z = 2y$ as an integral.

Soln: ⓪ Draw picture

↳ View from $x = 0$ plane



① Find intersection:

$$x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$$

So they intersect over the cylinder

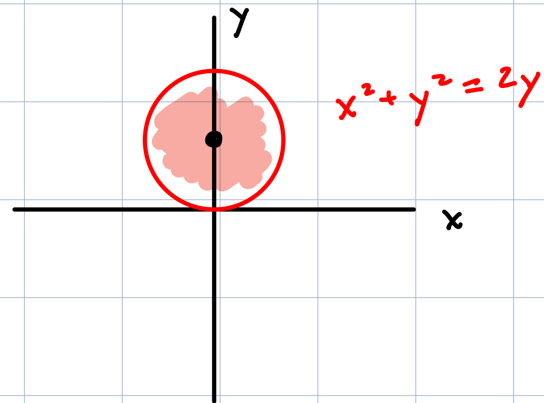
② Reduce to integral problem

↳ SA under $z = 2y$ is SA of $f = x^2 + y^2$

over D = inside of $\{(x,y) \mid x^2 + y^2 = 2y\}$.

$= \{x^2 + y^2 \leq 2y\}$

③ Draw D



④ Find polar bounds:

$D = \{ r \leq 2\sin(\theta) \,,\ 0 \leq \theta \leq \pi \}$

⑤ Set up integral and solve

$$SA = \iint_D \sqrt{1+(2x)^2+(2y)^2}\; dA$$

$$= \int_0^{\pi}\int_0^{2\sin(\theta)} r\cdot\sqrt{1+4r^2}\; dr\, d\theta$$

$u = 1+4r^2$

$du = 8r\,dr$

$$= \int_0^{\pi}\int_?^? \frac{1}{8}\sqrt{u}\; du\, d\theta$$

$$= \int_0^{\pi} \left(\frac{1}{8}\cdot\frac{2}{3}\cdot(1+4r^2)^{3/2} \right)\Big|_0^{2\sin(\theta)} d\theta$$

$$= \int_0^{\pi} \frac{1}{12}\cdot\left((1+16\sin^2\theta)^{3/2} - 1 \right) d\theta$$

$$= \; ?$$

Fact:

1. $f \geq g \Rightarrow \iint_D f \, dA \geq \iint_D g \, dA$

2. $\iint_D 1 \, dA = \text{Area}(D)$

2. If $m \leq f \leq M$, then

$$m \cdot \text{Area}(D) \leq \iint_D f \, dA \leq M \cdot \text{Area}(D)$$