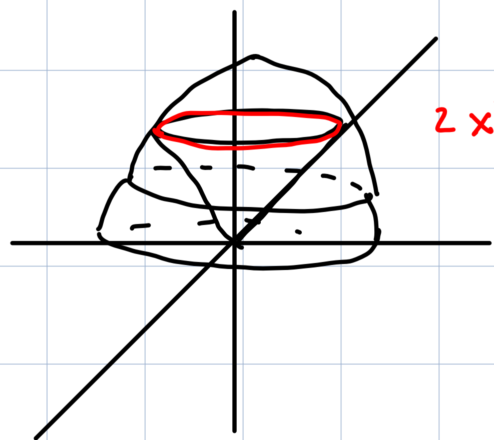
Lecture # 4

Title: Applications of double integrals - Probability

Section: Stewart 8.5, 15.4

Warm up: Find the volume above $z = \sqrt{x^2+y^2}$ and below $x^2+y^2+z^2 = 1$ as an integral in polar coordinates

Solution: ① Draw picture



② Phrase as integral

Volume = volume below $x^2+y^2+z^2 = 1$ over D

"minus" volume below $z = \sqrt{x^2+y^2}$ over D where $D = \{ x^2 + y^2 \leq 1/2 \}$

$\Rightarrow \text{Volume} = \iint_D \sqrt{1 - x^2 - y^2}\, dA - \iint_D \sqrt{x^2+y^2}\, dA$

③ Convert to polar

↳ $D = \{ r \leq 1/4 \}$

↳ $\text{Volume} = \int_0^{2\pi} \int_0^{1/4} r\sqrt{1 - r^2} - r^2 \, dr\, d\theta$

Remark:

- What is the probability that a random women is between 5' and 5' 2" ?
- How would we model and compute such a thing?
- Note all possible heights range over an interval of real numbers.

Definition: A continuous random variable X is a collection of possible outcomes that range over an interval of real numbers

Definition: The probability that the random variable is between a and b is $P(a \leq X \leq b)$

Example:

X = heights of women in inches

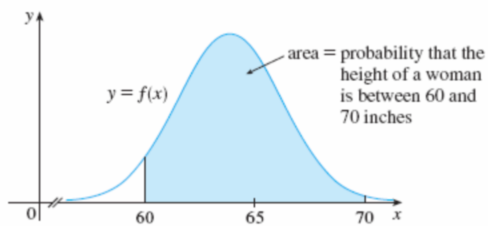
$P(60 \le X \le 70)$ = probability a random women has height between 60 and 70 inches.

Definition: The probability density function f of a continuous random variable X is the function that satisfies:

$$\int_a^b f(x)\,dx = P(a \le X \le b)$$

Example:

Probability density function for the height of an adult female



- Every probability density function satisfies

  1) $f(x) \geq 0$

  ↳ no such thing as "neg." likelihood

  2) $\int_{-\infty}^{\infty} f \, dx = 1$

  ↳ probability that any possibility occurs is 1

Example:

- $$f = \begin{cases} 0 & , \ |x| > 1 \\ |x| & , \ |x| \leq 1 \end{cases}$$

- $P(-5 \leq X \leq 0) = 1/2$

Question: What is average height of people on zoom meeting

Answer: Sum of all heights / # of people

Question: How could we approximate this w/out knowing exact heights, but just ranges where people fall.

Soln:

$$
\begin{aligned}
\text{Average} \approx\ & (\%\text{ people between } 4-5\text{ ft}) \cdot 4.5\text{ ft} \\
& + (\%\text{ people between } 5-6\text{ ft}) \cdot 5.5\text{ ft} \\
& + (\%\text{ people between } 6-7\text{ ft}) \cdot 6.5\text{ ft}
\end{aligned}
$$

Remark:

- What is the average value of X that occurs?
- We approximate it like before
- ↳ Divide $(-\infty, \infty)$ into n subintervals $[x_{i-1}, x_i]$ of widths Δx_i

↳ Average $\approx \sum_{i=1}^{n} x_i^* \cdot f(x_i^*) \cdot \Delta x_i$

for x_i^* sampled from $[x_{i-1}, x_i]$

↳ $\approx P(a \leq X \leq b)$

"=" % of occurrances in $[x_{i-1}, x_i]$

"=" % of women between 60"- 70" tall

↳ $\approx$ value of random variable over $[x_{i-1}, x_i]$

= "average" height among women in range is between 60"- 70".

↳ As # intervals goes to ∞, this becomes Reimann sum for $f(x) \cdot x$.

Definition: The mean of a pdf f is

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Warning: μ is <u>not</u> the average value of f as we saw in 15.1

Example:

- X = time you wait for professor to email you back
- $$f(t) = \begin{cases} 0 & , t \leq 0 \\ e^{-ct} & , t \geq 0 \end{cases}$$

When is f a pdf?

What is $P(0 \leq X \leq 5)$?

" " , " " μ ?

Solution:

- Must have $1 = \int_{-\infty}^{\infty} f \, dt$

$$\Rightarrow \quad 1 = \int_0^\infty e^{-tc} \, dt$$

$$= \left(-1/c \cdot e^{-tc}\right) \Big|_0^\infty$$

$$= 1/c$$

$$\Rightarrow \quad 1 = c$$

- $P(0 \le X \le 5) = \int_0^5 e^{-t} \, dt = -e^{-t} \Big|_0^5 = 1 - e^{-5}$
- $\mu = \int_0^\infty t \cdot e^{-t} = 1$ (integration by parts)

Definition: Let X, Y be two continuous random variables.

↳ X = height, Y = age

↳ Think of as points (X, Y) in plane $\mathbb{R}^2$

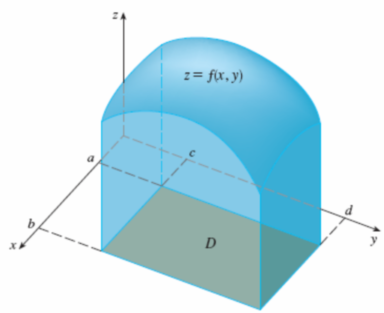
The joint density function of X and Y is the function f that satisfies

$$P(a \leq X \leq b,\ c \leq Y \leq d) = \int_c^d \int_a^b f \, dx \, dy$$

Picture:

Figure 7

The probability that X lies between a and b and Y lies between c and d is the volume that lies above the rectangle $D = [a, b] \times [c, d]$ and below the graph of the joint density function.



As before

i) $f \geq 0$

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \, dx \, dy = 1$

Example: Spse

$$f(x,y) = \begin{cases} C(x+2y) & , \quad 0 \leq x, y \leq 10 \\ 0 & \quad \text{else} \end{cases}$$

For what value C is f a pdf?

What is $P(X \leq 7, Y \geq 2)$?

Solution:

- Must have

$$1 = \int_0^{10}\int_0^{10} c(x+2y)\,dx\,dy$$

$$= c\int_0^{10} \left(x^2/2 + 2xy\right)\Big|_0^{10} dy$$

$$= c\int_0^{10} (50 + 20y)\,dy$$

$$= c\left(50y + 10y^2\right)\Big|_0^{10}$$

$$= c \cdot 1500$$

$$\Rightarrow \frac{1}{1500} = c$$

- $P(x \leq 7,\ y \geq 2)$

$$= \int_2^{10}\int_0^{7} c(x+2y)\,dx\,dy$$

$$= c\int_2^{10} \left(x^2/2 + 2yx\right)\Big|_0^{7} dy$$

$$= c\int_2^{10} 49/2 + 14y \ dy$$

$$= c\cdot\left(\frac{49}{2}y + 7y^2\right)\Big|_2^{10} = \text{etc}\ldots$$

Definition: X and Y are independent random variables if the joint density function is the product of their individual pdf's.

$$\hookrightarrow \quad f(x,y) = \underbrace{f_1(x)}_{=\text{pdf for } X} \cdot \underbrace{f_2(y)}_{=\text{pdf for } Y}$$

Remark:

$$P(a \leq X \leq b,\ c \leq Y \leq d)$$

$$= \int_a^b \int_c^d f_1(x) \cdot f_2(y)\, dy\, dx$$

some "constant"

$$= \int_a^b f_1(x) \int_c^d f_2(y)\, dy\, dx$$

$$= P(c \leq Y \leq d) \cdot \int_a^b f_1(x)\, dx$$

$$= P(c \leq Y \leq d) \cdot P(a \leq X \leq b)$$

Definition: The X-mean is (expected value of X)

$$\mu_X = \iint x \cdot f(x,y)\, dA$$

The Y-mean is (" " " Y)

$$\mu_Y = \iint y \cdot f(x,y)\, dA$$

Exercise: Spse X and Y have joint density function

$$f(x,y) = \begin{cases} \int_0^1 \int_0^1 9 \cdot x^2 \cdot y^2 & , \quad 0 \le x, y \le 1 \\ 0 & , \quad \text{else} \end{cases}$$

Compute the expected values of X and Y.

Solution:

- $$\begin{aligned} \mu_X &= \int_0^1 \int_0^1 x \cdot 9 x^2 y^2 \, dx \, dy \\ &= \int_0^1 (9x^3y^3/3)\Big|_0^1 \, dx \\ &= \int_0^1 3x^3 \, dx \\ &= (3/4 \cdot x^4)\Big|_0^1 \\ &= 3/4 \end{aligned}$$

- $\mu_Y = 3/4$