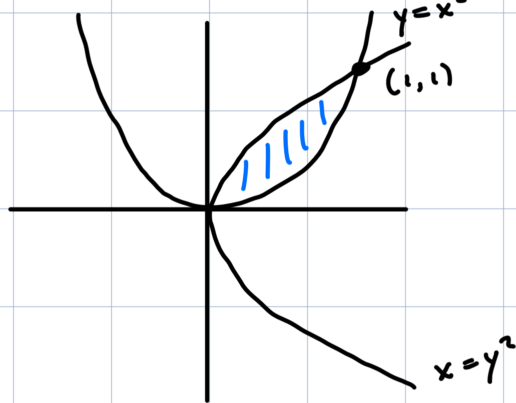
Lecture # 3

Title: Double integrals in polar coordinates

Section: Stewart 15.3

Warmup: Find the volume under the plane $3x + 2y - z = 0$ and above the region enclosed by $y = x^2$ and $x = y^2$

Solution: ① Draw D



② Bounds: $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$

③ Set up integral and solve

$$\iint_D 3x+2y\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 3x+2y\, dy\, dx$$

$$= \int_0^1 \left(3xy + y^2\right)\Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(3x^{3/2} + x - 3x^3 - x^4\right) dx$$

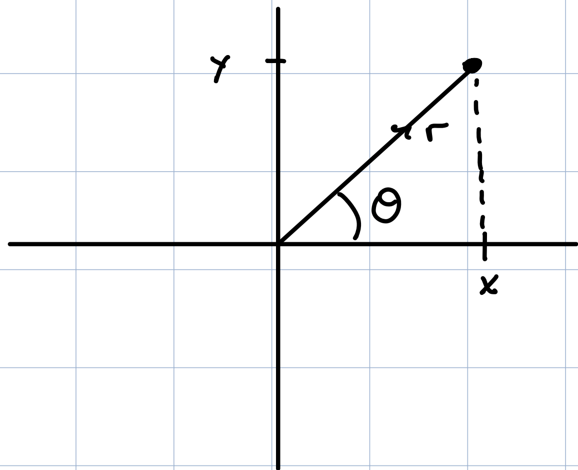
$$= \left(\frac{6}{5}x^{5/2} + \frac{x^2}{2} - \frac{3}{4}x^4 - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \frac{6}{5} + \frac{1}{2} - \frac{3}{4} - \frac{1}{5}$$

$$= 3/4$$

Review:

- We can describe points in $\mathbb{R}^2$ via Cartesian <u>or</u> polar coordinates



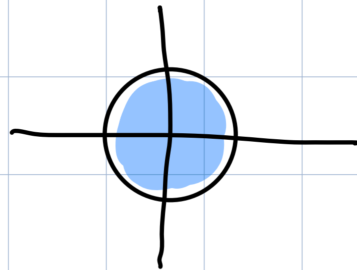
(x, y) = Cartesian

(r, θ) = polar

- Polar to Cartesian:
  - $x = r\cos(\theta)$
  - $y = r\sin(\theta)$
  - $r^2 = x^2 + y^2$

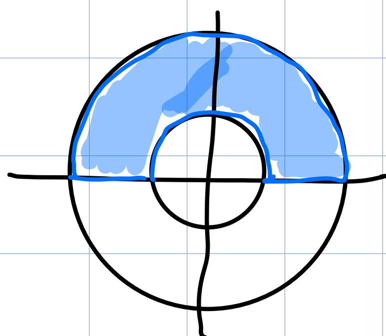
Remark: Describe regions in polar coordinates

- $\{(x,y) \mid x^2+y^2 \leq 1\} = \{(r,\theta) \mid r^2 \leq 1\}$



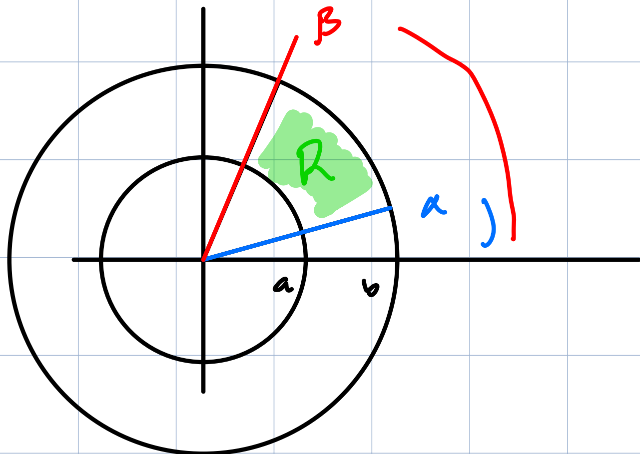
- $\{(x,y) \mid 1 \leq x^2+y^2 \leq 4,\ y \geq 0\}$

  $= \{(r,\theta) \mid 1 \leq r^2 \leq 4,\ 0 \leq \theta \leq \pi\}$



Remark: • R = polar rectangle

$$= \{(r,\theta) \mid a \le r \le b,\ 0 \le \alpha \le \theta \le \beta \le 2\pi\}.$$



• $\text{Area}(R) = b^2 \cdot \pi \cdot \left(\frac{\beta-\alpha}{2\pi}\right) - a^2 \pi \cdot \left(\frac{\beta-\alpha}{2\pi}\right)$

$$= (b^2 - a^2)\cdot(\beta-\alpha)/2$$

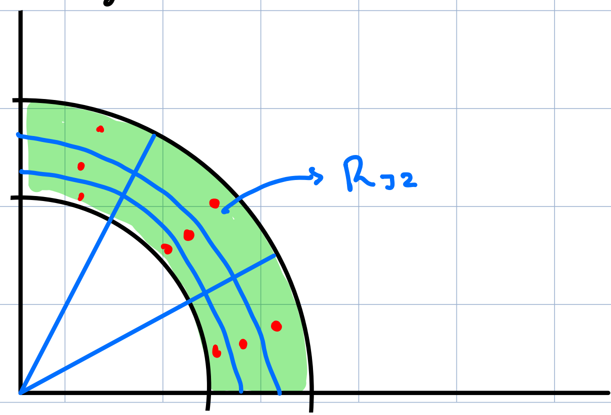
$$= r_{mid} \cdot \Delta r \cdot \Delta\theta$$

↳ $r^{mid} = (b+a)/2$ = middle radius in circle

- Divide R into smaller polar rectangles
  - ↳ Divide $[a,b]$ up as n subintervals $[r_{i-1}, r_i]$ of length $(b-a)/2$
  - ↳ Divide $[\alpha, \beta]$ up as n subintervals $[\theta_{i-1}, \theta_i]$ of length $(\beta - \alpha)/2$

Set $R_{ij} = \{ r_{i-1} \leq r \leq r_i \ , \ \theta_{i-1} \leq \theta \leq \theta_i \}$

↳ $r_i^{mid} = \dfrac{(r_i + r_{i-1})}{2}$, $\theta_j^{mid} = \dfrac{\theta_i + \theta_{i-1}}{2}$ are samples in R_{ij}



Fact: $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b r \cdot f(r\cos(\theta), r\sin(\theta))\,dr\,d\theta$

Proof: $\iint_R f(x,y)\,dA$

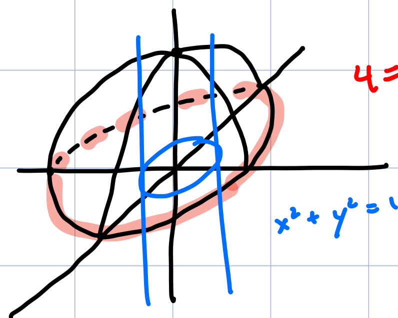
$= \lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^n f(r_i^{mid} \cdot \cos(\theta_j^{mid}), r_i^{mid}\sin(\theta_j^{mid})) \cdot \text{Area}(R_{ij})$

$= \lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^n f(r_i^{mid} \cdot \cos(\theta_j^{mid}), r_i^{mid}\sin(\theta_j^{mid}))\, r_i^{mid} \cdot \Delta r_i \cdot \Delta \theta_i$

$= \int_\alpha^\beta \int_a^b r \cdot f(r\cos(\theta), r\sin(\theta))\,dr\,d\theta$

Example: Find the volume between $z=0$ and $z=4-x^2-y^2$ and $x^2+y^2=1$

Solution: ① Sketch a picture



② Phrase as integral: Volume $= \iint_D 4-x^2-y^2 \, dx\,dy$

where $D = \{ 1 \leq x^2+y^2 \leq 4 \}$

$= \{ 1 \leq r \leq 2 \}$

(3) Describe using polar change of coordinates

$$\iint_D 4 - x^2 - y^2 \, dx\, dy = \int_0^{2\pi} \int_1^2 r \cdot (4 - r^2)\, dr\, d\theta$$

$$= 2\pi \left(2r^2 - \frac{r^4}{4}\right)\Big|_1^2$$

$$= 2\pi \left(8 - 2 - 4 + \frac{1}{4}\right)$$

$$= 2\pi (2 + 1/4)$$

$$= \pi (4 + 1/2)$$

Fact :

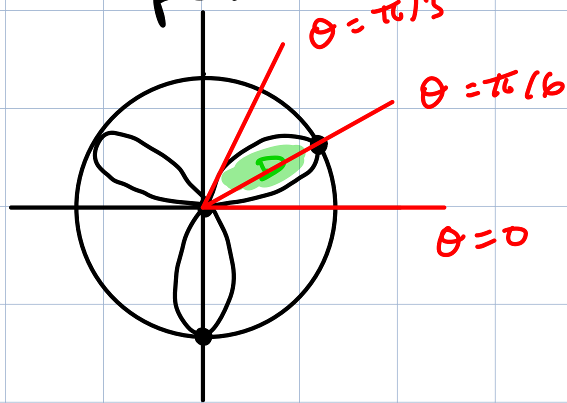
If $D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$,

then

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} r \cdot f(r\cos(\theta), r\sin(\theta))\, dr\, d\theta$$

Example: Find area of one loop of the rose $r = \sin(3\theta)$

Solution: ① Draw picture



$r = 0$ when

$\theta = 0, \frac{\pi}{3}$

② Set up polar bounds

$\hookrightarrow\ 0 \le r \le \sin(3\theta)$

$\hookrightarrow\ 0 \le \theta \le \frac{\pi}{3}$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ Defines D

③ Set up integral and solve

$$\text{Area}(D) = \iint_D 1 \, dA$$

$$= \int_0^{\pi/3} \int_0^{\sin(3\theta)} r \, dr \, d\theta$$

$$= \int_0^{\pi/3} \left(r^2/2\right)\Big|_0^{\sin(3\theta)} d\theta$$

$$= \int_0^{\pi/3} \sin^2(3\theta)/2 \, d\theta$$

$$= \frac{1}{4}\int 1 - \cos(6\theta) \, d\theta$$

$$= \frac{1}{4}\left(-\frac{\sin(6\theta)}{6} + \theta\right)\Big|_0^{\pi/3}$$

$$= \pi/12$$

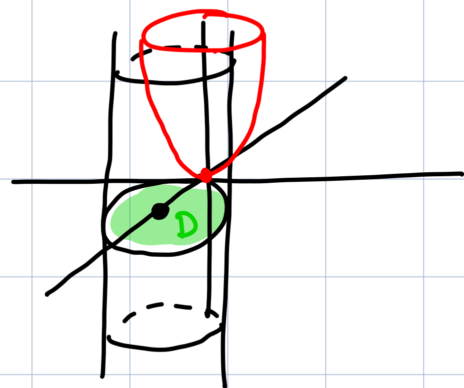
$$\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

Example: Find the volume below $z = x^2 + y^2$, above $z = 0$, and inside $x^2 + y^2 = 2x$

Solution: ① Draw picture

$$x^2 + y^2 = 2x$$
$$\Updownarrow$$
$$(x-1)^2 + y^2 = 1$$



② Phrase as integral:

Volume $= \iint_D x^2 + y^2 \, dA$ where

$$D = \{ (x-1)^2 + y^2 \le 1 \}$$
$$= \{ x^2 - 2x + y^2 \le 0 \}$$

③ Convert to polar coordinates and solve

$$D = \{ (r,\theta) \mid r^2 \leq 2r\cos(\theta) \}$$

$$= \{ (r,\theta) \mid r \leq 2\cos(\theta), -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2} \}$$

$$\Rightarrow \text{Volume} = \int_{-\frac{\pi}{2}}^{\pi/2} \int_{0}^{2\cos(\theta)} r^3 \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4} \right) \Big|_{0}^{2\cos(\theta)} dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^4(\theta) \, d\theta$$

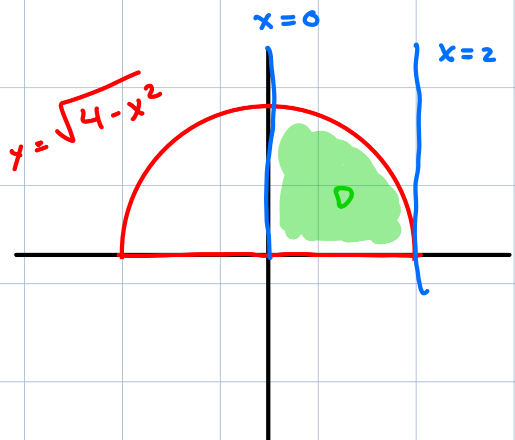
use double angle formula 2x.

= etc.

Example: Evaluate the iterated integral by converting to polar coordinates:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$$

Soln: ⓪ Draw picture of D



① Express D in polar

$$D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \pi/2\}$$

② Subsitute $x = r\cos(\theta)$, $y = r\sin(\theta)$, and mult. integrand $(e^{-x^2-y^2})$ by r and int. over new bounds

$$\int_0^{\pi/2}\int_0^2 e^{-r^2}\cdot r\, dr\, d\theta$$

$u = r^2$

$du = 2r\,dr$

$$= \frac{\pi}{4}\cdot\int_0^4 e^{-u}\,du$$

$$= \frac{\pi}{4}\cdot\left(-e^{-u}\right)\Big|_0^4$$

$$= \frac{\pi}{4}\cdot\left(-e^{-4}+1\right)$$