Lecture # 2

Title: Double Integrals over general regions

Section: Stewart 15.2

Warmup: Compute the average value of

$$f(x,y) = \frac{x}{y} + \frac{y}{x}$$

over $[1,2] \times [2,4] = R$

Thm: $$\iint_R f \, dA = \int_c^d \Big(\underbrace{\int_a^b f(x,y)\,dx}_{} \Big)\, dy$$

integral of x w/ y held constant

$$= \int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy}_{} \Big)\, dx$$

integral of y w/ x held constant

Defn: $$\text{Ave}(f \text{ over } R) = \frac{1}{\text{Area}(R)} \cdot \iint_R f(x,y)\, dA$$

Solution:

$$\text{Average} = \frac{1}{\text{Area}(R)} \cdot \int_1^2 \int_2^4 \frac{x}{y} + \frac{y}{x} \, dy \, dx$$

$$= \frac{1}{2} \int_1^2 \left(x \ln(y) + y^2/2x \right) \Big|_{y=2}^{4} dx$$

$$= \frac{1}{2} \int_1^2 \left(x(\ln(2)) + 6/x \right) dx$$

$$= \frac{1}{2} \left(\frac{x^2 \ln(2)}{2} + 6 \cdot \ln(x) \right) \Big|_{x=1}^{2}$$

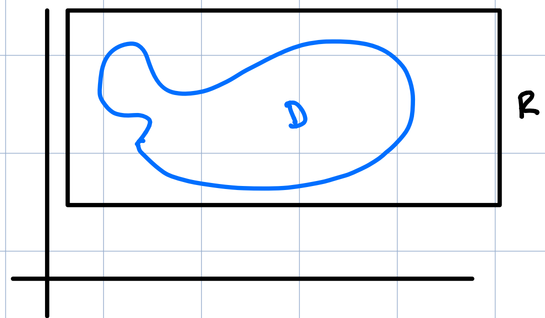
$$= \frac{1}{2} \left(\frac{\ln(2)}{2} \cdot 3 + 6 \ln(2) \right)$$

$$= \ln(2) \cdot (3 + 3/4)$$

Notation:

- D = region in plane
- R = rectangle containing D
- $f(x,y)$ = function

Picture:



Question: How can we compute the volume of f over D?

Definition:

$$F(x,y) = \begin{cases} f(x,y) & , (x,y) \text{ in } D \\ 0 & , \text{else} \end{cases}$$

Picture:

Figure 3

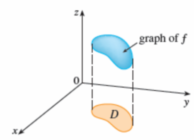
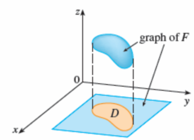


Figure 4



Definition:

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$$

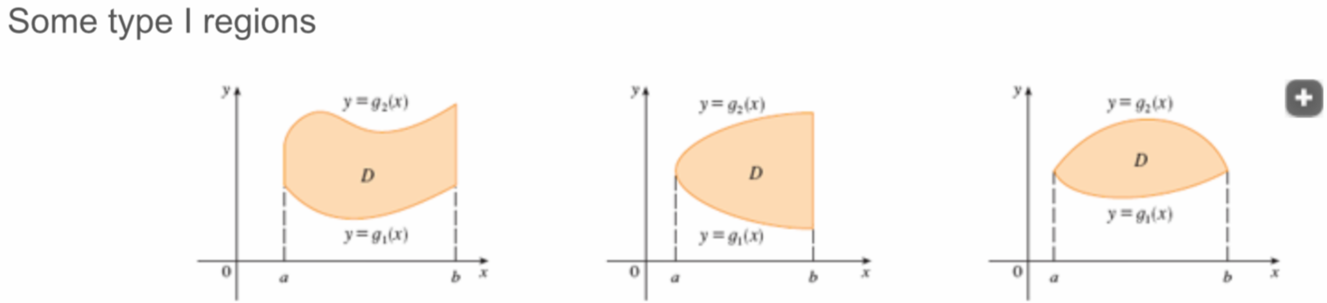
Remark: $\iint_D f(x,y)\,dA$ = volume of solid that lies below $z = f(x,y)$ and above D.

Definition: D is a region of type I if

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

for functions g_1, g_2 in x.

Picture:



Remark:

- $\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$
- If $g_1(x) \geq y$ or $g_2(x) \leq y$, then $F(x,y) = 0$

$$\Rightarrow \int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

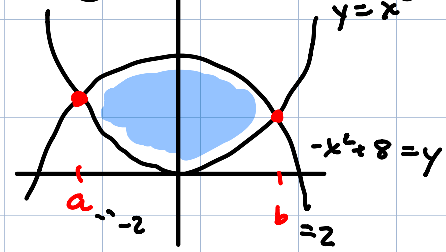
view as constants

Fact: If D is of type I, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Example: Evaluate $\iint_D (x^2+2)\,dA$ where D is bounded by $y = x^2$ and $y = -x^2+8$

① Draw



Solution: ① Solve for x bounds.

This is where $y = x^2$, $y=-x^2+8$ intersect

$$\Rightarrow \quad x^2 = -x^2+8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

② Set up the integral

$$\iint_D (x^2+2)\,dA = \int_{-2}^{2}\int_{x^2}^{-x^2+8} (x^2+2)\,dy\,dx$$

③ Integrate w/ x held constant

$$\int_{-2}^{2}\int_{x^2}^{-x^2+8} (x^2+2)\,dy\,dx$$

$$= \int_{-2}^{2} (yx^2+2y)\Big|_{x^2}^{-x^2+8}\,dx$$

$$= \int_{-2}^{2} (x^2+2)(-2x^2+8)\,dx$$

④ Solve remaining integral

$$\int_{-2}^{2} (x^2+2)(-2x^2+8)\,dx$$

$$= \int_{-2}^{2} -2x^4 + 4x^2 + 16 \,dx$$

$$= \left(-2x^5/5 + 4x^3/3 + 16x\right)\Big|_{-2}^{2}$$

$= \text{etc.}$

Definition: D is a region of type I if

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

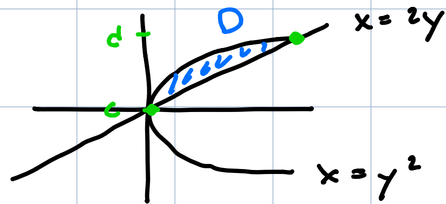
for functions h_1, h_2 in y.

Fact: If D is of type II, then

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Example: Find the volume below $z = 3x^2 + y^2$ above the region D bounded by $x = 2y$, $x = y^2$

Solution: ⓪ Draw D



① Solve for y bounds

$$2y = y^2 \Rightarrow y = 0 \text{ or } y = 2$$

② Set up integral

$$\iint_D 3x^2 + y^2 \, dx\, dy = \int_0^2 \int_{y^2}^{2y} 3x^2 + y^2 \, dx\, dy$$

③ Integrate w/ y constant

$$\int_{y^2}^{2y} 3x^2 + y^2 \, dx = \left(x^3 + xy^2 \right)\Big|_{y^2}^{2y} \, dx$$

$$= 8y^3 + 2y^3 - y^6 - y^4$$

④ Integrate what is left

$$\int_0^2 -y^6 - y^4 + 10y^3 \, dy$$

$$= \left(\frac{-y^7}{7} - \frac{y^5}{5} + \frac{10y^4}{4} \right)\Big|_0^2$$

$$= \frac{-128}{7} - \frac{32}{5} + \frac{160}{4}$$

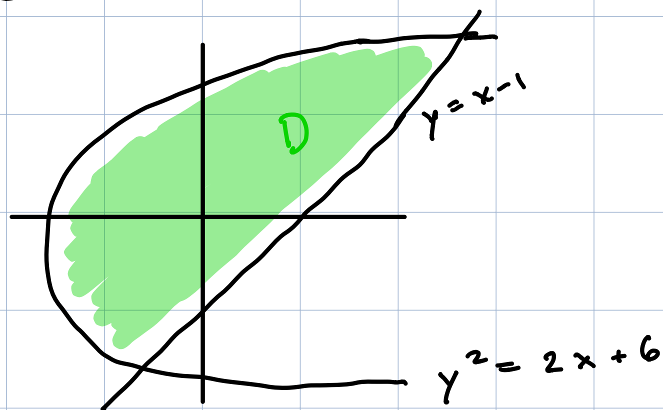
$$= 536/35$$

Exercise: Evaluate $\iint_D xy \, dA$ where is bounded by

$y = x - 1$ and $y^2 = 2x + 6$

Solution: ⓪ Draw D

$$\frac{y^2 - 6}{2} = x$$



① Realize as Type II

$\rightsquigarrow \quad x = y^2/2 - 3$ and $x = y + 1$

② Solve for y bounds

$\Rightarrow y^2/2 - 3 = y + 1 \Rightarrow y^2 = 2y + 8$

$\Rightarrow \quad y = -2, 4$

③ Set up integral

$$\iint_D xy \, dA = \int_{-2}^{4} \int_{y^2/2-3}^{y+1} xy \, dx \, dy$$

④ Integrate w/ y constant

$$\int_{y^2/2-3}^{y+1} xy \, dx = \left(\frac{x^2 y}{2} \right) \Big|_{y^2/2-3}^{y+1}$$

$$= (y+1)^2 y/2 - (y^2/2-3)^2 y/2$$

⑤ Finish integral

$$\int_{-2}^{4} (y+1)^2 y/2 - (y^2/2-3)^2 y/2 \, dy$$

$$= \frac{1}{2} \int_{-2}^{4} \frac{-y^5}{4} + 4y^3 + 2y^2 - 8y \, dy$$

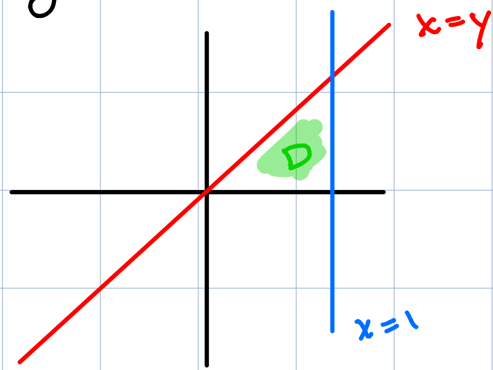
$$= \frac{1}{2} \left(\frac{-y^6}{24} + y^4 + \frac{2}{3} y^3 - 4y \right) \Big|_{-2}^{4}$$

$$= 36$$

Example: Evaluate $\int_0^1 \int_y^1 \cos(x^2)\, dx\, dy$

Solution: ① Change order of integration!

② Draw region D



③ Solve for new bounds

$$\iint_D \cos(x^2)\, dy\, dx = \int_0^1 \int_0^x \cos(x^2)\, dy\, dx$$

④ Solve new integral

$$\int_0^1 \int_0^x \cos(x^2)\,dy\,dx$$

$$= \int_0^1 \left(y \cdot \cos(x^2)\right)\Big|_0^x \,dx$$

$$= \int_0^1 x \cdot \cos(x^2)\,dx$$

$u = x^2$

$du = 2x\,dx$

$$= \int_0^1 \tfrac{1}{2}\cos(u)\,du$$

$$= \frac{1}{2}\sin(u)\Big|_0^1$$

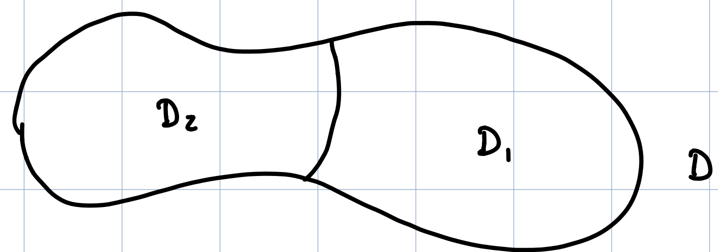
$$= \frac{1}{2}\sin(1)$$

Properties of integrals

Fact: If $D = D_1 \cup D_2$ where D_1 only meets D_2 only along boundaries, then

$$\iint_D f \, dA = \iint_{D_1} f \, dA + \iint_{D_2} f \, dA$$

Picture:



Remark: Sometimes you may need to divide D up into regions to integrate.