Lecture # 22

Title : Complex line integrals

Theorem: (Cauchy-Riemann Equations)

① If $f'(z_0)$ exists, then

$$(\star) \qquad \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

and $f'(z_0) = u_x(z_0) + i\, v_x(z_0)$

② If $u_x, v_x, u_y, v_y$ are continuous and satisfy $(\star)$ near $z_0$, then $f'(z_0)$ exists and

$$f'(z_0) = u_x(z_0) + i\, v_x(z_0)$$

Warm-up: Consider $f(z) = e^y \cdot e^{-ix}$. Determine where $f$ is holomorphic and give its value at these points.

Soln: $f(z) = e^y \cos(-x) + i e^y \sin(-x)$

$u_x = e^y \sin(-x)$

$v_x = -e^y \cos(-x)$

$u_y = e^y \cos(-x)$

$v_y = e^y \sin(-x)$

Need $u_x = v_y$, $u_y = -v_x$

$\Rightarrow e^y \sin(-x) = e^y \sin(-x)$ ✓

$e^y \cos(-x) = -(-e^y \cos(-x))$

$\Rightarrow f'(z) = e^y \sin(-x) + i(-e^y \cos(-x))$

Defn: An arc in $\mathbb{C}$ is a function

$$w(t) = x(t) + i\,y(t) \quad \text{w/} \quad a \le t \le b$$

The derivative of $w$ is

$$\frac{d}{dt}(w(t)) = w'(t) = x'(t) + i\,y'(t)$$

$\hookrightarrow$ Just differentiate normally wrt $t$.

Ex :

- $w(t) = Re^{it} = R\cos(t) + iR\sin(t)$

  $w'(t) = -R\sin(t) + iR\cos(t)$

  $= iRe^{it}$

- $w(t) = t^2 + it$

  $w'(t) = 2t + i$

- $w(t) = \exp(t^2 - i\cos(it))$

  $w'(t) = \exp(t^2 - i\cos(it)) \cdot (2t - \sin(it))$

Defn: The integral along a path $w(t)$ is

$$\int_a^b w(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$$

↪ Integrate real and imaginary parts separately

Example: $\displaystyle\int_0^{\pi/4} e^{it}\, dt = \int_0^{\pi/4} \cos(t)\, dt + i \int_0^{\pi/4} \sin(t)\, dt$

$$= \left(\sin(t) - i\cos(t)\right)\Big|_0^{\pi/4}$$

$$= \sqrt{2}/2 + i\left(1 - \frac{\sqrt{2}}{2}\right)$$

Fact: $\displaystyle\int_a^b w'(t)\, dt = w(b) - w(a)$

Ex:

$$\int_0^{\pi/4} e^{it}\,dt = \left(\frac{e^{it}}{i}\right)\Big|_0^{\pi/4} =$$

$$= \left(\frac{e^{i\pi/4}}{i} - \frac{1}{i}\right)$$

$$= \left(e^{i\pi/4}/i + i\right)$$

$$= \sqrt{2}/2 + i\left(1 - \sqrt{2}/2\right)$$

Ex: Compute $\int_0^1 (1+it)^2\,dt$.

Soln:

$$\int_0^1 (1+it)^2\,dt$$

$$= \left(\frac{-i}{3}(1+it)^3\right)\Big|_0^1$$

$$= \frac{-i}{3}\left((1+i)^3 - 1\right)$$

$$= \frac{-i}{3}\left(-3+2i\right)$$

$$= \frac{2}{3} + i$$

$(1+i)(1+i)(i+1)$

$(1+2i-1)(i+1)$

$-2+2i$

Defn: A contour $C$ is a piecewise smooth arc $w(t)$ in $\mathbb{C}$

Defn: Spse $w(t)$ traces out the contour $C$.
Let $f(z)$ be a fcn.
The contour integral of $f$ along $C$ is

$$\int_C f(z)\,dz = \int_a^b f(w(t)) \cdot w'(t)\,dt$$

$$= \int_a^b \text{Real}\,dt + i\int_a^b \text{imag.}\,dt$$

Warning: "$\cdot$" is multiplication of complex numbers, not the dot product!

Rem: $\int_C f\, dz = -\int_{-C} f\, dz$

Ex: $\int_C \frac{dz}{z}$ where $C$ is top half of unit circle.

Soln:
- $w(t) = e^{it}$ w/ $0 \le t \le \pi$
- $\int_0^{\pi} ie^{it}/e^{it}\, dt = \int_0^{\pi} i\, dt = \pi i$

Ex:

$f(z) = y - x - i3x^2$

$C$ is straight-line from $0$ to $1+i$.

Soln:

- $w(t) = t + it$ for $0 \leq t \leq 1$
- $w'(t) = 1 + i$
- $\int_C f \, dz = \int_0^1 (t - t - i3t^2)(1+i) \, dt$

$$= \int_0^1 3t^2 - i3t^2 \, dt$$

$$= \left( (1-i) t^3 \right) \Big|_0^1$$

$$= 1 - i$$

**Theorem:** If $f$ is holomorphic over a domain bounded by a simple* closed contour $C$, then

Cauchy - Goursat Theorem

$$\int_C f \, dz = 0$$

* Doesn't repeat path.

**Proof:**

$$\int_a^b f(w(t)) \cdot w'(t) \, dt$$

$$= \int_a^b \big(u(w(t)) + i v(w(t))\big) \cdot \big(x'(t) + i y'(t)\big) \, dt$$

$$= \int_a^b (u \cdot x' - v \cdot y') \, dt + i \int_a^b (v \cdot x' + u \cdot y') \, dt$$

* Dot prod w/ $\mathbb{C} \simeq \mathbb{R}^2$.

$$= \int_a^b (u, v) \cdot w' \, dt + i \int_a^b (v, u) \cdot w'(t) \, dt$$

Green!

$$= \iint_D v_x - u_y \, dA + i \int_a^b u_x - v_y \, dA$$

CR Equs.

$$= 0$$

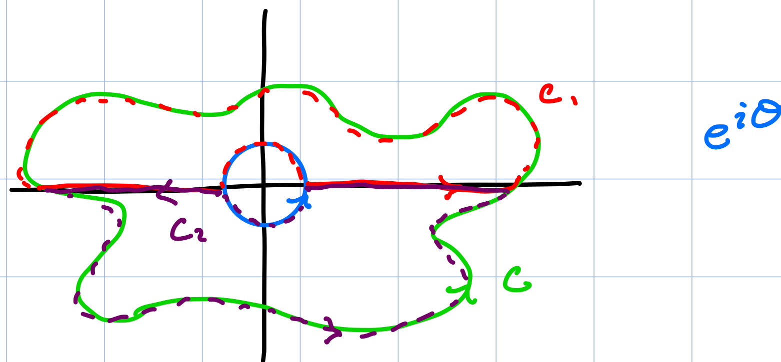
Ex: If $C$ is a closed curve that does not enclose $z=1$, then

- $\int_C \cos^2(z^2)\,dz = 0$
- $\int_C \sin(e^z)\,dz = 0$
- $\int_C z^2/(z-1)\,dz = 0$

Ex:

Spse $C$ is any simple closed curve that goes once around the origin counter clockwise. Show that $\int_C \frac{dz}{z} = 2\pi i$.

Soln:

- 

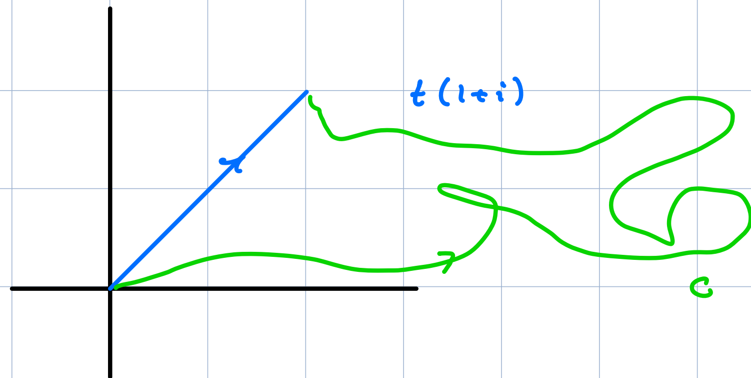

- Cauchy - Goursat $\Rightarrow \int_{c_1} \frac{dz}{z} = 0 = \int_{c_2} \frac{dz}{z} = 0$

$\Rightarrow \int_{c_1} \frac{dz}{z} + \int_{c_2} \frac{dz}{z} = \int_C \frac{dz}{z} - \int_{e^{i\theta}} \frac{dz}{z} = 0$

$\Rightarrow 2\pi i = \int_{e^{i\theta}} \frac{dz}{z} = \int_C \frac{dz}{z}$

Ex: Compute $\int_C z^2\,dz$ where $C$ is any arc from $0$ to $1+i$.

Soln: •




• CG $\Rightarrow$ $0 = \int_{-t(1+i)} z^2\,dz + \int_C z^2\,dz$

$$\Rightarrow \int_C z^2\,dz = \int_0^1 (t(1+i))^2 (1+i)\,dt$$

$$= \frac{1}{3}\left((1+i)t\right)^3 \Big|_0^1$$

$$= \frac{1}{3}(1+i)^3$$

Theorem: If $f$ is holomorphic over a domain bounded by a simple closed contour $C$, then

$$\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = f^{(n)}(z_0)$$

Ex: Spse $C = \{|z|=1\}$.

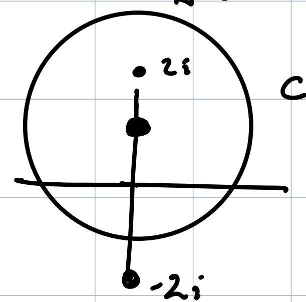
$$\int_C \exp(2z)/z^4\,dz = \frac{2\pi i}{6}\cdot 8 = 8\pi i/3$$

Ex: Compute $\int_c \cos(z)/z^3+9z \, dz$ where $C = \{ |z| = 1 \}$.

Soln: $$\int_c \left(\cos(z)/(z^2+9)\right)/z = 2\pi i \left( \frac{\cos(0)}{(0)^2+9} \right)$$

$$= 2\pi i / 9$$

Ex: $\int_c \frac{dz}{z^2+4}$ where $C = \{ |z - i| = 2 \}$.




Soln: $$\int_c \frac{1/(z+2i)}{z-2i} dz$$

$$= 2\pi i \cdot 1/4i$$

$$= \pi/2$$

**Theorem:** Every complex polynomial of degree $n$ has $n$ roots counting multiplicity

**Proof:**

- Spse $p(z)$ is a polynomial of degree $n$.
- If $p(z) \neq 0$, then $1/p(z)$ is holomorphic for all $z$ in $\mathbb{C}$.
- Consider the curve $C$ w/ $w(t) = R \cdot e^{it}$ for $0 \leq t \leq 2\pi$.

$$\frac{1}{p(0)} = \frac{1}{2\pi i} \int_C \frac{1/p(z)}{z} \, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{p(Re^{it})} \, dt$$

- But $|p(Re^{it})| \longrightarrow \infty$ as $R \longrightarrow +\infty$

  $\Rightarrow$ $1/p(Re^{it}) \longrightarrow 0$ as $R \longrightarrow +\infty$

  $\Rightarrow$ $2\pi i/p(0) = 0$, a contradiction.

- $\Rightarrow$ $1/p(z)$ is not holomorphic

  $\Rightarrow$ $p(z)$ has a root. $\square$

Remark: Please do course evaluations!