Lecture # 21

Title: Limits, Derivatives, and Cauchy-Reimann Equns.

Warm-up: 1 Let $z = 1 + i$, $w = 6 - i$. Compute

i) $z - w$

ii) $\bar{z} \cdot w$

iii) w / z

iv) $|w|$

v) Write z as $re^{i\theta}$ for some r and θ

Rem: We can write $f(x+iy) = u(x,y) + v(x,y)\,i$ as real and imaginary parts

Ex:

- $f(x+iy) = x^2 + xy\sin(x)\,i$
- $f(x+iy) = (x+iy)^2 = (x^2-y^2) + i\,(2xy)$
- $f(z) = z/z+1$

$$= \frac{z(\bar{z}+1)}{(z+1)(\bar{z}+1)}$$

$$= \frac{z\bar{z}+z}{z\bar{z}+z+\bar{z}+1}$$

$$= \frac{x^2+y^2+x}{x^2+y^2+2x+1} + i\,\frac{y}{x^2+y^2+2x+1}$$

* Fact: $\overline{z+w} = \bar{z} + \bar{w}$

Defn: The limit as $z \to z_0$ of $f(z)$ is w_0 if

$$\lim_{|z-z_0|\to 0} |f(z) - w_0| = 0$$

We write $\lim_{z\to z_0} f(z) = w_0$

Rem: The limit need not exist! $f(z) \to \pm\infty$, $1/0$, multiple cpx #s...

Ex: • $f(z) = i\bar{z}/2$, what is $\lim_{z\to 1} f(z)$?

Soln: As $x+iy = z \to 1 \Rightarrow x \to 1, y \to 0$

$$\Rightarrow f(z) = i(x-iy)/2$$

$$= y/2 + ix/2$$

$$\to i/2$$

Fact:

$$f(x+iy) = u(x,y) + iv(x,y)$$

$$z_0 = x_0 + iy_0 \ , \quad w_0 = u_0 + iv_0$$

Then $\lim_{z \to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$$

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$$

Ex:

- $f(z) = z/\bar{z}$, what is $\lim_{z \to 0} f(z)$?

Soln: $f(x+iy) = z^2/|z|^2$

$$= \frac{x^2-y^2}{x^2+y^2} + i\,\frac{2xy}{x^2+y^2}$$

As $(x,0) \longrightarrow (0,0)$

$f(x+i0) \longrightarrow 1$

As $(0,y) \longrightarrow (0,0)$

$f(0+iy) \longrightarrow -1$

$\Rightarrow$ Limit DNE

Defn: f is continuous at z_0 if as $z \to z_0$, we have $f(z) \to f(z_0)$.

↳ As $x+iy \to x_0+iy_0$, $u(x,y) \to x_0$

" " " " , $v(x,y) \to y_0$

Cor: f is continuous if and only if u, v are cont.

Defn: The derivative of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

↳ f is differentiable at z_0 if $f'(z_0)$ exists.

Ex:

- $f(z) = z$

$\hookrightarrow \lim\limits_{z \to z_0} \dfrac{z - z_0}{z - z_0} = 1$

$\Rightarrow f'(z) = 1$ (and thus exists)

Ex: • $f(z) = \bar{z}$

$$\hookrightarrow \lim_{\substack{x \to x_0 \\ y = y_0}} \frac{x - iy_0 - x_0 + iy_0}{x + iy_0 - x_0 - iy_0}$$

$$= \lim_{x \to x_0} \frac{x - x_0}{x - x_0}$$

$$= 1$$

$$\neq -1$$

$$= \lim_{\substack{x = x_0 \\ y \to y_0}} \frac{-iy + iy_0}{+iy - iy_0}$$

$$= \lim_{\substack{x = x_0 \\ y \to y_0}} \frac{x_0 - iy - x_0 + iy_0}{x_0 + iy - x_0 - iy_0}$$

$f'(z)$ DNE

Fact:

1) $\frac{d}{dz}(c) = 0$

2) $\frac{d}{dz}(z^n) = n \cdot z^{n-1}$

3) $\frac{d}{dz}(f(z) \cdot g(z)) = f'(z) \cdot g(z) + g'(z) \cdot f(z)$

4) $\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z)$.

5) $\frac{d}{dz}(f \circ g(z)) = f'(g(z)) \cdot g'(z)$.

6) $\frac{d}{dz}(\cos(z)) = -\sin(z)$

7) $\frac{d}{dz}(\sin(z)) = \cos(z)$

8) $\frac{d}{dz}(e^z) = e^z$

Ex:

- $\frac{d}{dz}(\sin(z^2)) = 2z\cos(z^2)$
- $\frac{d}{dz}((2z^2+i)^5) = 5(2z^2+i)^4(4z)$
- $\frac{d}{dz}(xy + ixy^3) =$ Does it even exist?

Theorem: (Cauchy - Riemann Equations)

① If $f'(z_0)$ exists, then

$$(\star) \qquad \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

and $f'(z_0) = u_x(z_0) + i\, v_x(z_0)$

② If u_x, v_x, u_y, v_y are continuous and satisfy $(\star)$ near z_0, then $f'(z_0)$ exists and

$$f'(z_0) = u_x(z_0) + i\, v_x(z_0)$$

Ex:

- $f(x+iy) = z^2 = x^2 - y^2 + 2xy\,i$

We showed that $f'(z)$ exists (or it is easy to see)

$$\left.\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \end{aligned}\right\} \Rightarrow \begin{aligned} u_x &= 2x = v_y \\ u_y &= -2y = -v_x \end{aligned}$$

$$\Rightarrow f'(z) = 2x + i\,2y = 2z$$

- $f(x+iy) = x - iy$

$$u_x = 1 \neq -1 = v_y$$

$\Rightarrow f'(z)$ does not exist

Idea: Spse $z_0 = 0$ and $f'(0)$ exists.

$$\lim_{x \to 0} \frac{f(x+0i) - f(0)}{x}$$

$$= \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} + i \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= u_x(0,0) + i v_x(0,0)$$

$$\lim_{y \to 0} \frac{f(0+yi) - f(0)}{iy}$$

$$= \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{iy} + i \lim_{x \to 0} \frac{v(y,0) - v(0,0)}{iy}$$

$$= u_y(0,0)/i + v_y(0,0)$$

$$= -i u_y(0,0) + v_y(0,0)$$

$$\Rightarrow u_x = v_y, \quad u_y = -v_x \qquad \square$$

Ex:

$$f(z) = e^x e^{iy} = e^x \cos(y) + i e^x \sin(y)$$

Show that $f'(z)$ exists for all z.

- $u_x = e^x \cos(y) = v_y$
- $u_y = -e^x \sin(y) = -v_x$

$\Big\}$ CR-eqn hold $\Rightarrow f'(z)$ exists.

- $f'(z) = u_x + i v_x = e^x \cos(y) + i e^x \sin(y)$

Defn: If $f'(z)$ exists for z in $D \subseteq \mathbb{C}$, then f is said to be holomorphic over D.

Ex: Is $1/\bar{z}$ holomorphic away from 0?

- $\frac{1}{\bar{z}} = \frac{z}{|z|^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$

$$u_x = (y^2 - x^2)/(x^2+y^2)^2$$
$$v_y = (x^2 - y^2)/(x^2+y^2)^2$$

$\Rightarrow$ Not holomorphic!

Ex:

$f(z) = |z|^2$

$\hookrightarrow u(x,y) = x^2 + y^2$

$\hookrightarrow v(x,y) = 0$

$u_x = 2x \neq v_y \ , \ u_y = 2y \neq -v_x$

$\Rightarrow f'(z)$ DNE for any non-zero z.

$\hookrightarrow$ Not holomorphic.

Exercise: Find when $f'(z)$ exists and its value for

$$f(z) = x^2 + ixy$$

Soln:

$$\left.\begin{aligned} u_x &= 2x \\ v_x &= y \\ u_y &= 0 \\ v_y &= x \end{aligned}\right\}$$

CR-eqn hold when $x = 0$ and $y = 0$.

So $f'(0) = 0$ at the existence point.

Fact:

1) $\cos(z)$, $\sin(z)$, e^z are holomorphic

2) The composition of holomorphic fcns is holomorphic

3) Sum, product, (non-zero) division of holo. fcns is holomorphic

Ex:

1) $\sin(z^2)$ is holo

2) $(z^2+z)/(z-3)$ is holo when $z \neq 3$

3) $\exp\left(\dfrac{\cos(z)}{z-1}\right)$ is holo when $z \neq 1$.

Ex :

If $f(z)$ and $\overline{f(z)}$ are both analytic, then $f(z)$ is constant.

Proof :

- $f = u + iv$, $\bar{f} = u - iv$
- CR $\overset{f}{\Longrightarrow}$ $u_x = v_y$, $u_y = -v_x$

  $\overset{\bar{f}}{\Longrightarrow}$ $u_x = -v_y$, $u_y = v_x$

  $\Rightarrow$ u_x, u_y, v_x, v_y are zero
- $\Rightarrow$ u, v are constant

  $\Rightarrow$ f is constant. $\square$