Lecture # 20

Title:

- Divergence Theorem
- Complex numbers, polynomials, and functions

Divergence Theorem

* The surface has no boundary curves.

* closed means it completely bounds a solid.

Thm: (Divergence Theorem)

- Let E be a solid bounded by a closed* surface ∂E
- Spse ∂E is pos. oriented ($\hat{n}$ points outwards from E).
- Let F = vf whose comp. fcns have cont. partial derivatives in a region that contains E.

$$\iint_{\partial E} F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV$$

↳ Also called Gauss's Theorem.

↳ like the proof of Green's Theorem but w/ regions replaced by solids and curves replaced by surfaces.

Ex:

Compute $\iint_S F \cdot d\vec{S}$ where

↳ $F = (z, y, x)$

↳ $S =$ unit sphere

Soln:

- $$\iint_S F \cdot d\vec{S} = \iiint_{Ball} \nabla \cdot F \, dV$$
$$= \iiint_{Ball} 0 + 1 + 0 \, dV$$
$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$
$$= 4\pi/3$$

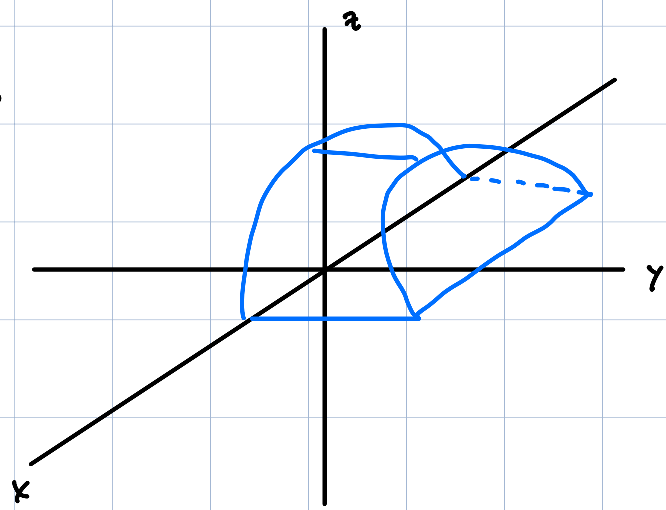
Ex: Compute $\iint_S F \cdot d\vec{S}$, where

↳ $F = (xy, y^2 + e^{xz^2}, \sin(xy))$

↳ S = surface bounding the solid E that is contained

by $z = 1 - x^2$, $z = 0$, $y = 0$, $y + z = 2$

Soln: • Draw:



xy, $y^2 + e^{xz^2}$, $\sin(xy))$

- $$\iint_S F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV$$

$$= \iiint_E 3y \, dV$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, dy \, dz \, dx$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \frac{3}{2}(2-z)^2 \, dz \, dx$$

$$= \int_{-1}^{1} \left(\frac{-1}{2}(2-z)^3 \right) \Big|_0^{1-x^2} dx$$

$$= \int_{-1}^{1} \left(\frac{-1}{2}(1+x^2)^3 - \frac{-1}{2} 8 \right) dx$$

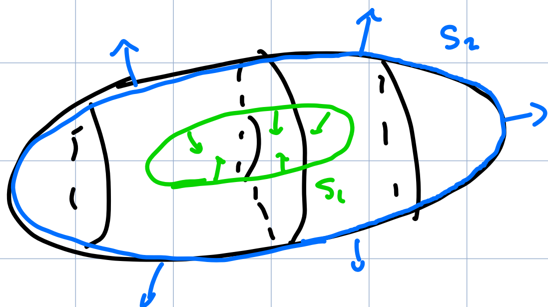
$$= \int_0^1 8 - (1+x^2)^3 \, dx$$

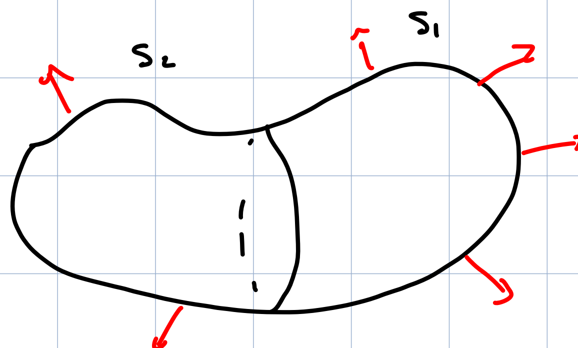
$= \text{etc}$

Rem:

$$\iiint_E \mathrm{Div}(F)\, dV$$
$$= \iint_{\partial E} F \cdot d\vec{S}$$
$$= \iint_{S_1} F \cdot d\vec{S} + \iint_{S_2} F \cdot d\vec{S}.$$

Picture:





Rem: If $\nabla \cdot F = 0$, then $\iint_{S_1} F \cdot d\vec{S} = -\iint_{S_2} F \cdot d\vec{S}$

$\hookrightarrow$ Amount flow in = Amount flow out

Fact: $\nabla \cdot F = 0$ if and only if $\nabla \times G = F$

Ex:

Compute $\iint_S F \cdot d\vec{S}$ where

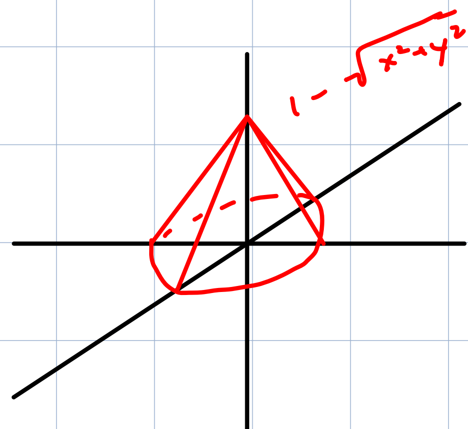
↪ $F = (xz, x^2, z^2)$

↪ S = surf on $z = 1 - \sqrt{x^2+y^2}$ above $z=0$

w/ normal dir. pointing towards

xy-plane

Soln: • Draw:



- $\iiint_E \nabla\cdot F \, dV = -\iint_S F\cdot d\vec{S} - \iint_{S_1} F\cdot d\vec{S}$

  where S_1 = unit disk in xy-plane w/ upwards orientation

- $\nabla\cdot F = 3z$

- $\iiint_E 3z \, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r} r\,3z \, dz\,dr\,d\theta$

$$= 2\pi \int_0^1 \frac{3}{2}(1-r)^2 r \, dr$$

$$= 3\pi \int_0^1 r - 2r^2 + r^3 \, dr$$

$$= 3\pi\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

- S_1: $\vec{r}(u,v) = (u,v,0)$, $\vec{r}_u \times \vec{r}_v = (0,0,1)^*$ up

  $\Rightarrow \iint_{S_1} 0\cdot d\vec{S} = 0$

  $\Rightarrow \iint_S F\cdot d\vec{S} = 3\pi\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$

Stokes like theorems in Dimension 3

$$\text{Functions} \xrightarrow{\nabla} \text{Vector fields} \xrightarrow[\text{curl}]{\nabla\times} \text{Vector fields} \xrightarrow[\text{div}]{\nabla\cdot} \text{Functions}$$

$$f\big|_{\partial C} \overset{FTLI}{=\!=} \int_C \nabla f \cdot d\vec{r}$$

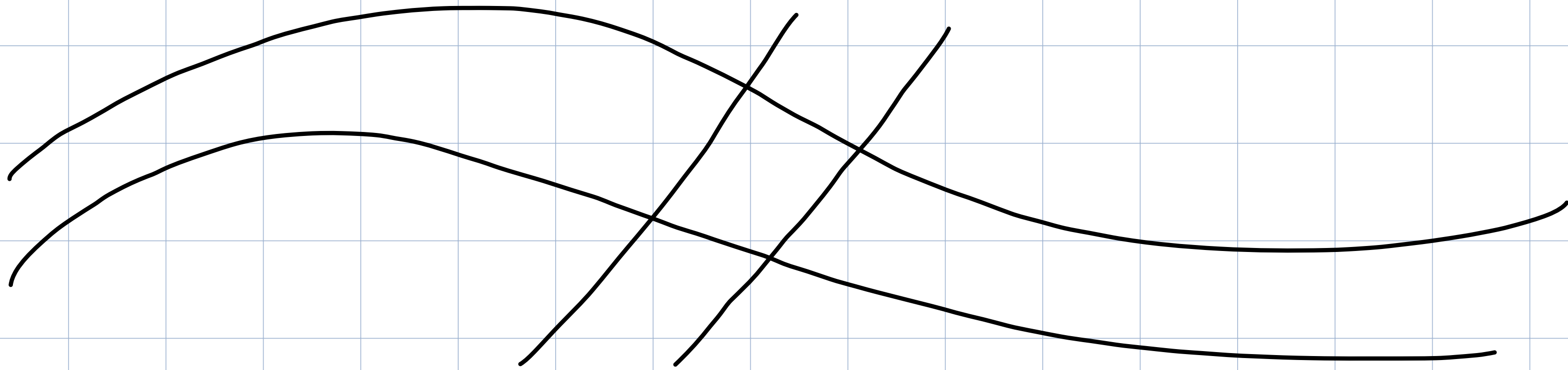
$$\int_{\partial S} F \cdot d\vec{r} = \iint_S \nabla\times F \cdot d\vec{S}$$

$$\iint_{\partial E} F \cdot d\vec{S} = \iiint_E \nabla\cdot F \, dV$$

- $\nabla\times(\nabla f) = 0$, $\nabla\cdot(\nabla\times F) = 0$

  ↳ Apply two operators in a row gives 0 !

- $F = \nabla f$ if and only if $\nabla\times F = 0$
- $F = \nabla\times G$ if and only if $\nabla\cdot F = 0$



Complex Analysis (Calculus w/ Complex Numbers).

Definition:

- A <u>real polynomial</u> is a fcn $f: \mathbb{R} \longrightarrow \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where each a_i is a real number.

- When $a_n \neq 0$, we say the <u>degree of f</u> is

$$\deg(f) = n$$

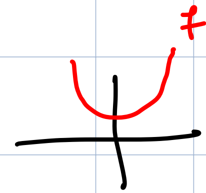
- If $f(x_0) = 0$, then we say x_0 is a <u>root</u> of f.

Example: $f(x) = x^{77} - 17x^{66} + 42x - 26$

↳ $\deg(f) = 77$

↳ $f(1) = 0 \Rightarrow 1$ is a root.

Remark:



- Not all real polynomials have real roots
- $f(x) = x^2 + 1$

  If $f(x) = 0$, then $0 = x^2 + 1 \Rightarrow x^2 = -1$.

  But the square of a real number is never negative

  $\Rightarrow$ f has <u>no</u> roots

- There just aren't enough real numbers.
- If $i = \sqrt{-1}$, then $f(i) = 0$ so f would have a root.
- Need to make sense of such numbers.

Definition: The complex numbers $\mathbb{C}$ is the set

$$\mathbb{C} = \{ (x,y) \text{ in } \mathbb{R}^2 \} = \{ x + iy \mid (x,y) \text{ in } \mathbb{R}^2 \}$$

$\hookrightarrow$ ie, a complex number is a formal sum $x + iy$ where x and y are real numbers.

$\hookrightarrow$ x is called the real part of $x + iy$

$\hookrightarrow$ y " " " imaginary " " "

Notation: We will often write $z = x + iy$ to denote a complex number.

Remark: We can add complex numbers

$$(x_0 + iy_0) + (x_1 + iy_1) = (x_0 + x_1) + i(y_0 + y_1)$$

$$\leadsto \quad (18 + 7i) + (-25 - 2i) = -7 + 5i$$

Remark: We can multiply complex numbers by requiring $i^2 = -1$

$$(x_0 + iy_0) \cdot (x_1 + iy_1)$$

$$= x_0 x_1 + i(x_0 y_1) + i(y_0 x_1) + i^2 y_0 y_1$$

$$= x_0 x_1 - y_0 y_1 + i(x_0 y_1 + x_1 y_0)$$

$$\leadsto \quad (2+i) \cdot (7 - 7i) = 14 - 14i + 7i - 7i^2 = 21 - 7i$$

Definition: The norm of a complex number $x+iy$ is

$$|x+iy| = \sqrt{x^2+y^2}$$

Defn: The complex conjugate of a complex number $x+iy$ is $x-iy$

Rem: • $|z|^2 = z \cdot \bar{z}$

Remark: If $|u+iv| \neq 0$, then we can divide $x+iy$ by $u+iv$

$$\frac{x+iy}{u+iv} \cdot \frac{u-iv}{u-iv}^{*} = \frac{x+iy}{u+iv} \cdot \frac{u-iy}{u-iv}$$

$$= \frac{(x+iy)\cdot(u-iv)}{u^2 \cancel{-iuv} + \cancel{iuv} - i^2 v^2}$$

* $u - iv = \overline{u+iv}$

$$= \frac{(x+iy)\cdot(u-iv)}{u^2+v^2}$$

$$= \frac{(x+iy)\cdot(u-iv)}{|u+iv|^2} \qquad (*)$$

We can make sense of $(*)$ since we can just scale the real and imaginary parts of numerator by the denominator, which is a real number

Remark: Just as we can talk about fcns from $\mathbb{R}$ to $\mathbb{R}$, we can talk about fcns from $\mathbb{C}$ to $\mathbb{C}$.

Definition: A fcn $f : \mathbb{C} \to \mathbb{C}$ is an assignment of a complex number z to the complex number $f(z)$.

↳ e.g. $f(z) = z^2 - 17$

Definition:

- A complex polynomial is a fcn $f: \mathbb{C} \to \mathbb{C}$ of the form

$$f(x) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

e.g. $f(z) = iz^2 - 17z + (1-i)$.

where each a_i is a complex number.

- When $a_n \neq 0$, we say the degree of f is

$$\deg(f) = n$$

- If $f(z_0) = 0$, then we say z_0 is a root of f.

Thm: Every complex polynomial of degree n has n roots (counting multiplicity).

Remark: • One way to define the fcn $e^x : \mathbb{R} \to \mathbb{R}$ is via taylor's series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots (2)(1).$

Remark: So a Taylor series is approximated by a seq. of polynomials. These Taylor series have to satisfy some "convergence" properties, ie, this infinite sum alway needs to converge to something finite.

↪ So some calculus is required to make this rigorous.

↪ The calculus also carries over to the complex case.

⇒ Use Taylor series w/ complex numbers.

Definition: The complex exponential fcn is the fcn $e^z : \mathbb{C} \to \mathbb{C}$ given by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Lemma: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for θ a real number.

Proof: We use the Taylor series for sin and cos and compute.

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$\underbrace{}_{n = \text{even}} \qquad \underbrace{}_{n = \text{odd}}$$

$$= \sum_{k=0}^{\infty} \frac{i^{2k}\,\theta^{2k}}{(2k)!} + \sum_{\ell=0}^{\infty} \frac{i^{2\ell+1}\,\theta^{2\ell+1}}{(2k+1)!}$$

$$(i)^{2k} = ((i)^2)^k = (-1)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\,\theta^{2k}}{(2k)!} + i\sum_{\ell=0}^{\infty} \frac{(-1)^\ell\,\theta^{2\ell+1}}{(2k+1)!}$$

$$= \cos(\theta) + i\sin(\theta)$$

$\square$

Corollary: $e^{i\pi} = -1$

Remark: • We identify $\mathbb{C}$ w/ $\mathbb{R}^2$ via $x+iy \longleftrightarrow (x,y)$

↪ Suggest polar form for cpx numbers

$$re^{i\theta} = r\cos(\theta) + i r\sin(\theta) \longleftrightarrow (r\cos(\theta), r\sin(\theta))$$

↪ gives geom intuition for $|\cdot|$

$|x+iy| = |(x,y)| =$ distance from (x,y) to the origin

$$|re^{i\theta}| = |r|$$

↪ gives geom intuition for $\bar{z}$.

$$\bar{z} = x - iy \longleftrightarrow (x,-y)$$

$\Rightarrow$ $\bar{\cdot}$ reflects $\mathbb{C}$ across real-axis (x-axis).