Lecture # 19

Title: Stokes's Theorem

Section: Stewart 16.8

Defn: Let S be an oriented surface w/ unit normal vector $\vec{n}$ and param. $\vec{r}(u,v)$. The surface integral of F over S is

* Some fcn.

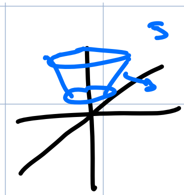
$$\iint_S F \cdot d\vec{S} = \iint_S F \cdot \vec{n} \, dS$$

$$= \iint_D F(\vec{r}(u,v)) \cdot \frac{(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_D F(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

Also called flux of F across S.

Warm-up: Compute $\iint_S F \cdot \vec{n}\, d\vec{S}$ where



$F = (-x, -y, z^3)$

$S =$ part of cone $z = \sqrt{x^2+y^2}$ between $z=1$ and $z=3$ w/ downward orientation.

Soln:

- $\vec{r}(u,v) = (u, v, \sqrt{u^2+v^2})$, $1 \le \sqrt{u^2+v^2} \le 3$
- $\vec{r}_u \times \vec{r}_v = \left(\frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \right)^*$ upwards
- $F(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) =$

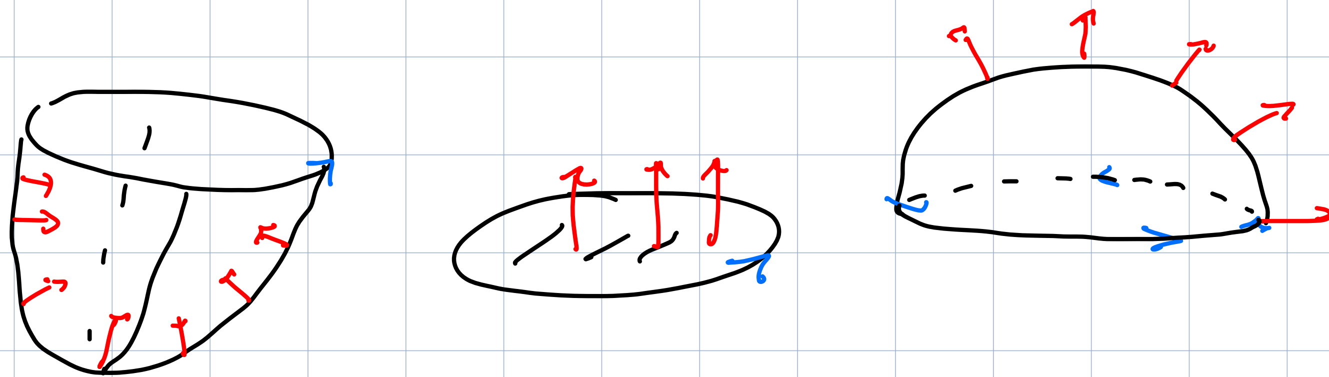
$$= (-u, -v, (u^2+v^2)^{3/2}) \cdot \left(\frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \right)$$

$$= \sqrt{u^2+v^2} + (u^2+v^2)^{3/2}$$

- $\iint_S F \cdot \vec{n}\, d\vec{S} = -\int_0^{2\pi} \int_1^3 r\,(r + r^3)\, dr\, d\theta =$ etc.

Defn: Let S be an oriented surface w/ boundary. The orientation of S induces pos. orientation of boundary curve.

Picture:



Rem: Use the right-hand rule.

Theorem: Let S = oriented surface that is bounded by closed, piecewise smooth boundary curve(s) ∂S w/ the pos. orientation.

Let F = vf on $\mathbb{R}^3$ whose component fcns have continuous partial derivatives on region that contains S.

$$\int_{\partial S} F \cdot d\vec{r} = \iint_S \nabla \times F \cdot d\vec{S}$$

Ex:

- Spse D = region in xy-plane.
- Spse $F = (P(x,y), Q(x,y), 0)$
- $\vec{r}(u,v) = (u,v,0)$ w/ (u,v) in S
- $\vec{r}_u = (1,0,0)$, $\vec{r}_v = (0,1,0)$
- $\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = (0, 0, Q_x - P_y)$
- $\iint_D Q_x - P_y \, dA = \iint_D (\nabla \times F) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$

$$= \iint_D (\nabla \times F) \cdot d\vec{S}$$

Stokes's Theorem

$$= \int_{\partial D} F \cdot d\vec{r}$$

* lies in xy-plane

$\Rightarrow$ This is Green's Theorem.

Ex:

Compute $\int_C F \cdot d\vec{r}$ where

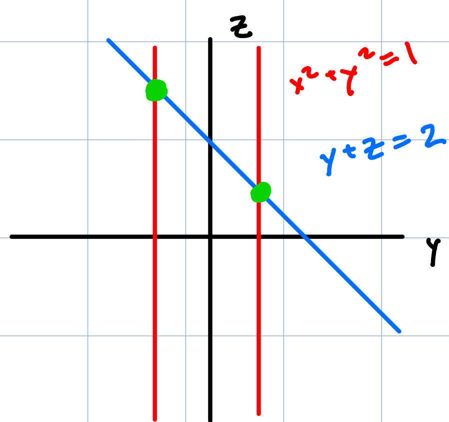
↳ $F = (-y^2, x, z^2)$

↳ C = intersection of plane $y + z = 2$ and cylinder $x^2 + y^2 = 1$.

Oriented counter clockwise from above

Soln:

- Draw:



- Intersection: Some ellipse

- $\int_C F \cdot d\vec{r} = \iint_S \nabla \times F \cdot d\vec{S}$, where S* is the graph of $f(x,y) = 2-y$ over disk of radius 1.
- $\vec{r}(u,v) = (u, v, 2-v)$

  $\vec{r}_u(u,v) = (1,0,0)$

  $\vec{r}_v(u,v) = (0,1,-1) \rightsquigarrow$ *downwards pointing!

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = (0,1,-1)$$

- $$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = (0, 0, 1+2y)$$

$$\bullet \quad \int_C F \cdot d\vec{r} = -\iint_S (0,0,1+2v) \cdot (0,1,-1)\, dA$$

$$= \iint_S 2v+1\, dA$$

$$= \int_0^{2\pi} \int_0^1 r\,(2r\sin(\theta)+1)\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}\sin(\theta) + \frac{1}{2}\, d\theta$$

$$= \pi$$

Rem: Trick is to find a simple bounding surface

Ex:

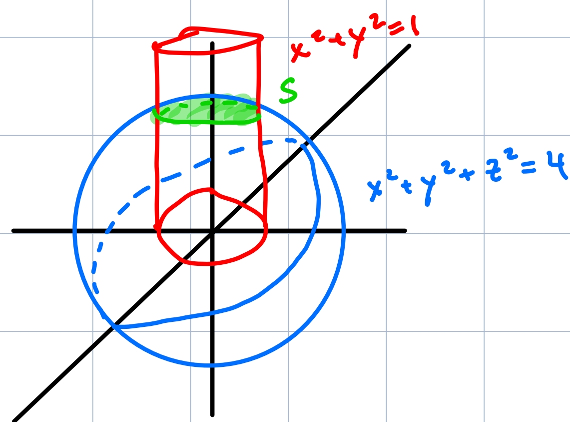
Compute $\iint_S \nabla \times F \cdot d\vec{S}$ where

↳ $F = (xz, yz, xy)$

↳ S = part of sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2+y^2=1$

Soln:

- Draw:



- Intersection: $x^2+y^2=1$ and $x^2+y^2+z^2=4 \Rightarrow z^2=3$

$\Rightarrow \partial S: \vec{r}(t) = (\cos(t), \sin(t), \sqrt{3})$

for $0 \le t \le 2\pi$

(xz, yz, xy)

- Stokes Thm:

$$\iint_S \nabla \times F \cdot d\vec{S} = \int_{\partial S} F \cdot d\vec{r}$$

$$= \int_0^{2\pi} \left(\sqrt{3}\cos(t), \sqrt{3}\sin(t), \cos(t)\sin(t)\right) \cdot (-\sin(t), \cos(t), 0)\, dt$$

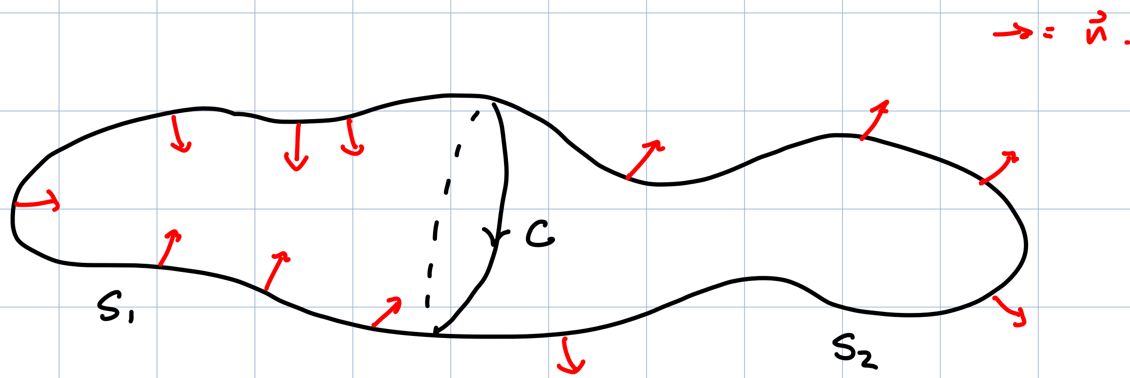
$$= \int_0^{2\pi} 0 \, dt$$

$$= 0$$

Rem: If S_1 and S_2 both bound C, then

$$\iint_{S_1} \nabla \times F \, d\vec{S} = \int_C F \cdot d\vec{r} = \iint_{S_2} \nabla \times F \, d\vec{S}$$

Picture:



Rem: Sometimes one bounding surface will be easier to compute w/ than another.

Thm: Spse $F = \nabla f$ w/ comp. fcns having continuous partial derivatives. Then F is conservative if and only if $\nabla \times F = 0$.

Proof:

- Previously, $\nabla \times (\nabla f) = 0$

  $\Rightarrow$ If F is conservative, then $\nabla \times F = 0$.

- Spse $\nabla \times F = 0$.

  Spse C = closed curve.

  There exists some surface S st $\partial S = C$.

  $$\Rightarrow \int_C F \cdot d\vec{r} = \iint_S \nabla \times F \cdot d\vec{S} = \iint_S 0 \, dS = 0$$

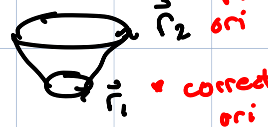
  $$\Rightarrow \int_C F \cdot d\vec{r} = 0 \text{ for all closed curves } C$$

  $\Rightarrow$ F is conservative. $\square$

Example: Compute $\iint_S \nabla \times F \, d\vec{S}$

$F = (-y, x, z^2)$

$S =$ part of cone $z = \sqrt{x^2+y^2}$ between $z=1$ and $z=3$ w/ downward orientation.



Soln:

- $\vec{r}_1(t) = (\cos(t), \sin(t), 1)$

  $\vec{r}_2(t) = (3\cos(t), 3\sin(t), 3)$

- $\vec{r}_1{}'(t) = (-\sin(t), \cos(t), 0)$

  $\vec{r}_2{}'(t) = (-3\sin(t), 3\sin(t), 0)$

- $F(\vec{r}_1) \cdot \vec{r}_1{}' = \sin^2 t + \cos^2 t = 1$

  $F(\vec{r}_2) \cdot \vec{r}_2 = 9$

- $\iint_S \nabla F \cdot d\vec{S} = + \int_0^{2\pi} 1 \, dt - \int_0^{2\pi} 9 \, dt = -16\pi$

Divergence Theorem

* The surface has no boundary curves.

Thm: (Divergence Theorem)

- Let E be a solid bounded by a closed* surface ∂E
- Spse ∂E is pos. oriented ($\hat{n}$ points outwards from E).
- Let F = vf whose comp. fcns have cont. partial derivatives in a region that contains E.

$$\iint_{\partial E} F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV$$

↳ Also called Gauss's Theorem.

↳ like the proof of Green's Theorem but w/ regions replaced by solids and curves replaced by surfaces.

Stokes like theorems in Dimension 3

$$\text{Functions} \xrightarrow{\nabla} \text{Vector fields} \xrightarrow[\text{curl}]{\nabla\times} \text{Vector fields} \xrightarrow[\text{div}]{\nabla\cdot} \text{Functions}$$

$$f\big|_{\partial C} \overset{\text{FTLI}}{=\!=} \int_C \nabla f \cdot d\vec{r}$$

$$\int_{\partial S} F \cdot d\vec{r} = \iint_S \nabla\times F \cdot d\vec{S}$$

$$\iint_{\partial E} F \cdot d\vec{S} = \iiint_E \nabla\cdot F \, dV$$

- $\nabla\times(\nabla f) = 0$, $\nabla\cdot(\nabla\times F) = 0$

  ↳ Apply two operators in a row gives 0 !

- $F = \nabla f$ if and only if $\nabla\times F = 0$
- $F = \nabla\times G$ if and only if $\nabla\cdot F = 0$???