Lecture # 18

Title: Surface Integrals of Vector Fields

Section: Stewart 16.7

Review:

The surface integral of $f(x,y,z)$ over S is

$$\iint_S f \, dS$$

$$= \iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_D f(x(u,v), y(u,v), z(u,v)) \cdot |\vec{r}_u \times \vec{r}_v| \, dA$$

where S has param eqn

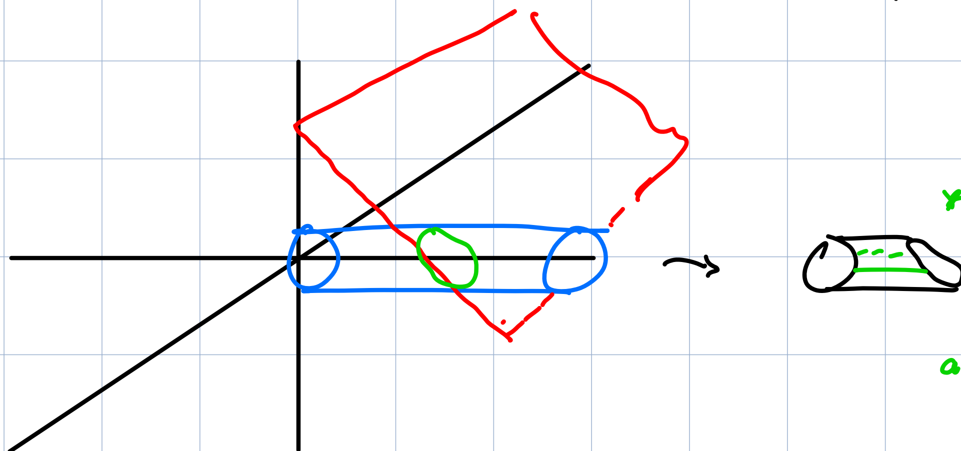
$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

w/ u, v in D.

Warmup: Give a double integral expression for $\iint_S e^x \sin(z)\, dS$ where S is the part of the cylinder $1 = x^2 + z^2$ that is between $z = 0$, $y = 0$, and $z = 2 - y$

Soln:

- Draw:



* Do top half above $z = 0$.

- $\vec{r}(u,v) = (\cos(u), v, \sin(u))$

  $0 \leq u \leq \pi$, $\quad 0 \leq v \leq 2 - \sin(u)$

- $\vec{r}_u = (-\sin(u), 0, \cos(u))$

  $\vec{r}_v = (0, 1, 0)$

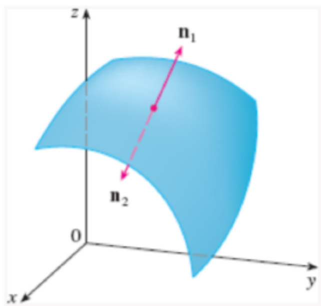
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -\sin(u) & 0 & \cos(u) \\ 0 & 1 & 0 \end{vmatrix} = (-\cos(u), 0, -\sin(u))$$

- $\iint_S xy \, dS = \int_0^{\pi} \int_0^{2-\sin(u)} e^{\cos(u)} \sin(\sin(u)) \, dv\, du.$

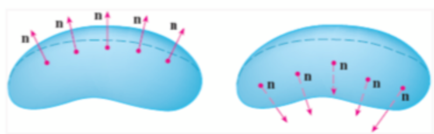
Orientations

Defn: A surface S is orientable if one can choose a normal vector at every point in S st the normal vectors vary continuously over S.

Picture:



The two orientations of an orientable surface



Remark: If S is orientable w/ param eqn $\vec{r}(u,v)$, then we have unit normal vectors:

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Ex:

$$\vec{r}(u,v) = \left(a\cos(u)\sin(v), a\sin(u)\sin(v), a\cos(v)\right)$$

$$\vec{r}_u \times \vec{r}_v = \left(a^2\sin^2(v)\cos(u), a^2\sin^2(v)\sin(u), a^2\sin(v)\cos(u)\right)$$

$$|\vec{r}_u \times \vec{r}_v| = a^2\sin(v)$$

$$\Rightarrow \vec{n} = \left(\sin(v)\cos(u), \sin(v)\sin(u), \sin(v)\cos(u)\right)$$

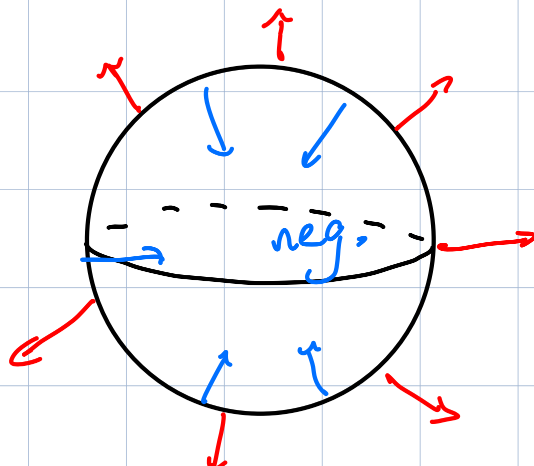
$$= (x, y, z)/a$$

$$= \vec{r}/a$$

Defn:

- If S bounds a solid E, then S is said to be <u>closed</u>.
- A positive orientation is one for which $\vec{n}$ points outwards from E
- " neg. " " " " " " " inward
  " "

Picture:



Surface Integrals of Vector Fields

Defn: Let S be an oriented surface w/ unit normal vector $\vec{n}$ and param. $\vec{r}(u,v)$. The surface integral of F over S is

* Some fcn.

$$\iint_S F \cdot d\vec{S} = \iint_S F \cdot \vec{n}\, dS$$

$$= \iint_D F(\vec{r}(u,v)) \cdot \frac{(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \iint_D F(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

Also called flux of F across S.

Ex :

Compute $\iint_S F \cdot d\vec{S}$ where

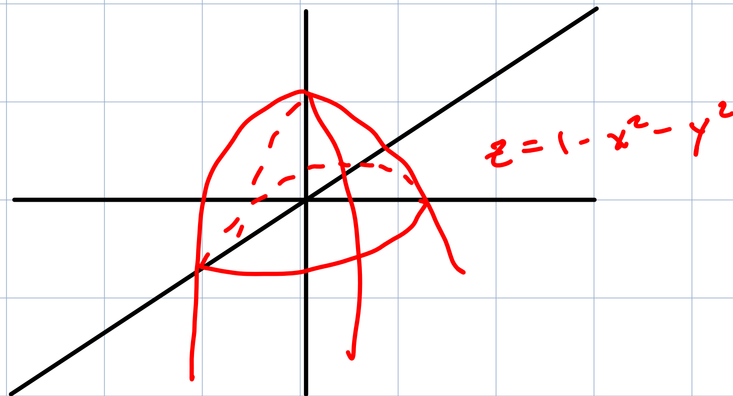
↳ $F(x,y,z) = (x, y, z)$

↳ S is the boundary of the solid E, where

E is solid enclosed by $z = 1 - x^2 - y^2$, $z = 0$

Soln :

- Draw :



- S_1 = paraboliod part

  S_2 = Disk part.

- $\vec{r}_1(u,v) = (u, v, 1 - u^2 - v^2)$

  $\vec{r}_2(u,u) = (u, v, 0)$

- $\vec{r}_{1u} \times \vec{r}_{1v} = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = (+2u, +2v, 1)$

This normal points outwards!

- $\iint_{S_1} F \cdot d\vec{S} = \iint_D (u, v, 1 - u^2 - v^2) \cdot (+2u, +2v, 1)\, dA$

$$= \iint_D 2u^2 + 2v^2 + 1 - u^2 - v^2 \;\; dA$$

$$= \iint_D u^2 + v^2 + 1 \qquad dA$$

$$= 2\pi \int_0^1 r^3 + r \quad dr$$

$$= 2\pi \left(\frac{1}{4} + \frac{1}{2} \right)$$

$$\bullet \quad \vec{r}_{2u} \times \vec{r}_{2v} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0, 0, 1)$$

$$\Rightarrow \iint_{S_2} F \cdot d\vec{S} = \iint_D (u, v, 0) \cdot (0, 0, 1) \, dA = 0$$

Rem:

- $\measuredangle$ between $F(\vec{r}(u,v))$ and $\vec{n}$ is $< 90^\circ$

  $\Rightarrow \quad F(\vec{r}(u,v)) \cdot \vec{n} \geq 0$

  $\Rightarrow \quad \iint_S F \cdot d\vec{S} \geq 0$

- " " " " " " $= 90^\circ$

  $\Rightarrow$ " $= 0$

  $\Rightarrow \quad \iint_S F \cdot d\vec{S} = 0$

- " " " " " " $> 90^\circ$

  $\Rightarrow$ " ≤ 0

  $\Rightarrow \quad \iint_S F \cdot d\vec{S} \leq 0$

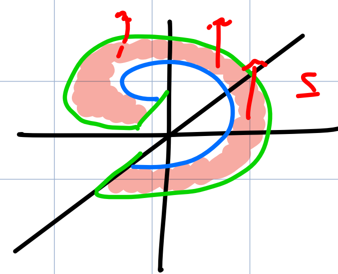
$\hookrightarrow$ $\iint_S F \cdot d\vec{S}$ measures total amount that F flows through the surface S in the direction of $\vec{n}$.

Exercise: Compute $\iint_S F \cdot d\vec{S}$ where

↳ S has param. $\vec{r}(u,v) = (v\cos(u),\ v\sin(u),\ u)$

w/ $0 \le u \le 2\pi$, $1 \le v \le 2$ and is oriented upwards

↳ $F(x,y,z) = (x, y, z)$



Soln:

- $\vec{r}_u = (-v\sin(u),\ v\cos(u),\ 1)$

  $\vec{r}_v = (\cos(u),\ \sin(u),\ 0)$

  $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -v\sin(u) & v\cos(u) & 1 \\ \cos(u) & \sin(u) & 0 \end{vmatrix} = (-\sin(u), \cos(u), -v)$

- $\vec{r}_u \times \vec{r}_v$ is <u>downwards</u> pointing.

- $\iint_S F \cdot d\vec{S}$

$$= -\int_0^{2\pi}\int_1^2 (v\cos u, v\sin u, u) \cdot (-\sin(u), \cos(u), -v) \; dv\,du$$

$$= -\int_0^{2\pi}\int_1^2 -uv \; dv\,du$$

$$= \int_0^{2\pi} \frac{u}{2}(3)$$

$$= 6\pi^2$$

Rem:

- $S: z = g(x,y) \leftrightarrow f^{-1}(0)$, where $f(x,y,z) = z - g(x,y)$
  $\Rightarrow \nabla f$ is normal vector to the surface.

- $\vec{r}(u,v) = (u, v, g(u,v))$
  $\vec{r}_u = (1, 0, g_u)$, $\vec{r}_v = (0, 1, g_v)$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & g_u \\ 0 & 1 & g_v \end{vmatrix}$$

$$= (-g_u, -g_v, 1)$$

$$= \nabla f(\vec{r}(u,v))$$

Ex: Set-up $\iint_S F \cdot d\vec{S}$ where S is the upwards oriented paraboloid $z = x^2 + y^2$ over the unit disk and $F = (x, 0, z)$.

Soln:

- $S = f^{-1}(0)$ where $f(x,y,z) = z - x^2 - y^2$
- $\vec{r}(u,v) = (u, v, x^2 + y^2)$ w/ u, y in unit disk.
- $\Rightarrow \vec{r}_u \times \vec{r}_v = \nabla f(\vec{r}(u,v))$

  $\nabla f = (-2x, -2y, 1)$

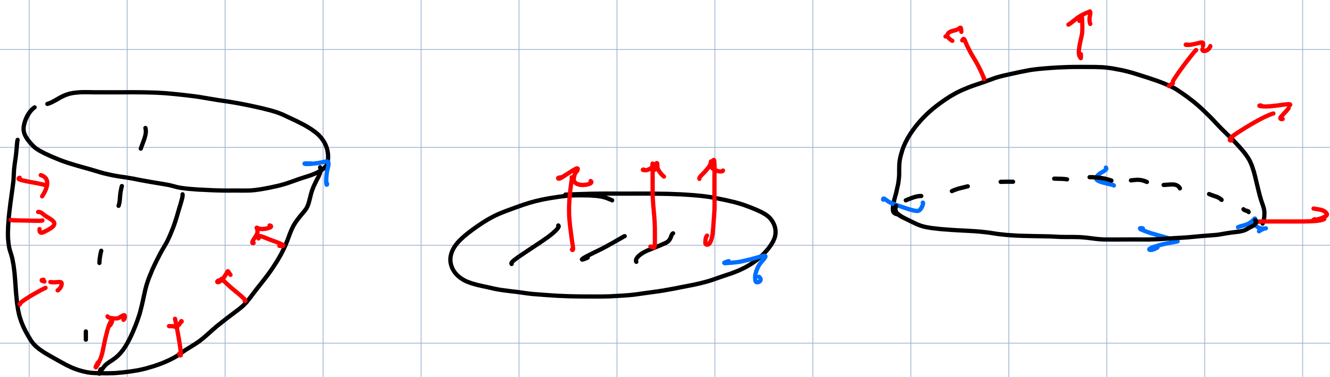
  $= (-2u, -2v, 1)$ * upwards
- $\iint_S F \cdot d\vec{S} = \iint_D (u, 0, u^2 + v^2) \cdot (-2u, -2v, 1)\, dA$

  $= \iint -2u^2 + u^2 + v^2 \, dA$

  $=$ etc.

Defn: Let S be an oriented surface w/ boundary. The orientation of S induces pos. orientation of boundary curve.

Picture:



Rem: Use the right-hand rule.

Theorem: Let S = oriented surface that is bounded by closed, piecewise smooth boundary curve(s) ∂S w/ the pos. orientation.

Let F = vf on $\mathbb{R}^3$ whose component fcns have continuous partial derivatives on region that contains S.

$$\int_{\partial S} F \cdot d\vec{r} = \iint_S \nabla \times F \cdot d\vec{S}$$

Stokes like theorems in Dimension 3

$$\text{Functions} \xrightarrow{\nabla} \text{Vector fields} \xrightarrow[\text{curl}]{\nabla\times} \text{Vector fields} \xrightarrow[\text{div}]{\nabla\cdot} \text{Functions}$$

$$f\big|_{\partial C} \overset{FTLI}{=\!=} \int_C \nabla f \cdot d\vec{r}$$

$$\int_{\partial S} F \cdot d\vec{r} = \iint_S \nabla \times F \cdot d\vec{S}$$

$$\int_? ?\, d? = \int_? \nabla \cdot ?\, d?$$

- $\nabla \times (\nabla f) = 0$, $\nabla \cdot (\nabla \times F) = 0$

  ↳ Apply two operators in a row gives 0 !

- $F = \nabla f$ if and only if $\nabla \times F = 0$
- $F = \nabla \times G$ if and only if $\nabla \cdot F = 0$???