Lecture # 17

Title: Surface integrals

Section: Stewart 16.7

Review: • Given $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

the tangent plane at $(x_0, y_0, z_0) = \vec{r}(u_0, v_0)$ is

spanned by

↳ $\vec{r}_u(u_0, v_0) = (x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0))$,

↳ $\vec{r}_v(u_0, v_0) = (x_v(u_0,v_0), y_v(u_0,v_0), z_u(u_0,v_0))$

It has normal vector $\vec{r}_u \times \vec{r}_v$.

Warm-up: Consider $\vec{r}(u,v) = (e^u \cos(v), e^u \sin(v), u^2)$

w/ $0 \le u \le 2$, $0 \le v \le \pi$.

Describe the associated surface and compute the normal vector at $(0, e, 1)$.

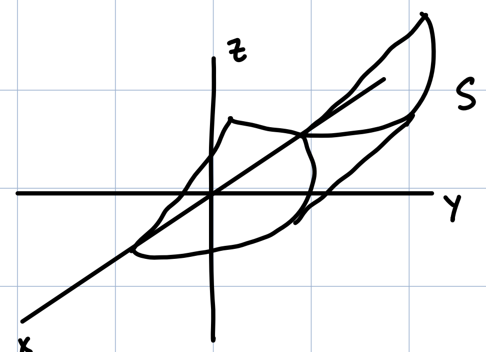
Soln:

* constant/fixed

- $\vec{r}(u_0, v) = (e^{u_0} \cos(v), e^{u_0} \sin(v), u_0^2)$

  ↳ Since $0 \le v \le \pi \Rightarrow \vec{r}(u_0, v)$ is a half circle of radius e^{u_0} raised u_0^2 above $z=0$ plane.

- Draw



- $\vec{r}_u(u,v) = \left(e^u\cos(v),\ e^u\sin(v),\ 2u\right)$

  $\vec{r}_v(u,v) = \left(-e^u\sin(v),\ e^u\cos(v),\ 0\right)$

- $\vec{r}(1,\pi/2) = (0,e,1)$

- $\text{normal} = (0,e,2)\times(-e,0,0)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & e & 2 \\ -e & 0 & 0 \end{vmatrix}$$

$$= (0,-2e,e^2)$$

- tangent plane is: $-2e(y-e) + e^2(z-1) = 0$

Surface Area

Notn:

- Spse $\vec{r}(u,v)$ has domain D in uv-plane.
- Divide D up into rectangles R_{ij}
- Spse $\text{Area}(R_{ij}) = \Delta u \cdot \Delta v$
- Sample (u_i^*, v_j^*).

Defn:

- $SA = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n}$ SA of $\vec{r}(R_{ij})$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n}$ SA of tangent plane at $\vec{r}(u_i^*, v_j^*)$ over rect. of size $\Delta u \cdot \Delta v$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n}$ SA of parallelogram spanned by $\Delta u \cdot \vec{r}_u(u_i^*, v_j^*)$ and $\Delta v \cdot \vec{r}_v(u_i^*, v_j^*)$

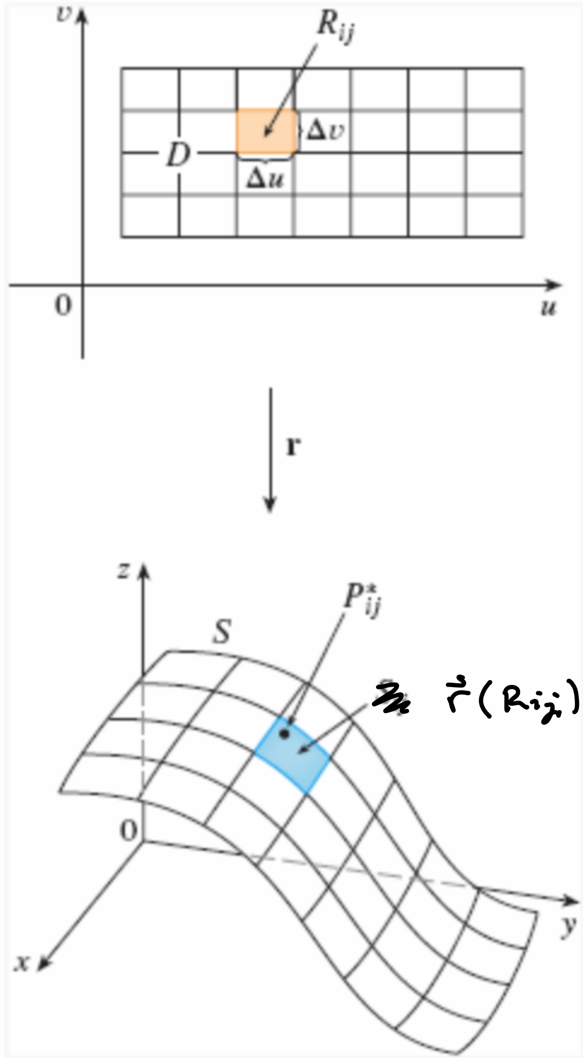
$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \Delta u \cdot \vec{r}_u(u_i^*, v_j^*) \times \Delta v \cdot \vec{r}_v(u_i^*, v_j^*) \right|$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \vec{r}_u(u_i^*, v_j^*) \times \vec{r}_v(u_i^*, v_j^*) \right| \cdot \Delta u \cdot \Delta v$

$= \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$

- This is the <u>surface area</u> of S when S is covered just once as (u,v) range over D.

Picture :



Ex: SA of sphere of radius a.

Soln:

- $\vec{r}(u,v) = \left(a\cos(u)\sin(v),\ a\sin(u)\sin(v),\ a\cos(v)\right)$

  w/ $0 \le \overset{\theta}{u} \le 2\pi$, $0 \le \overset{\phi}{v} \le \pi$

- $\vec{r}_u = \left(-a\sin(u)\sin(v),\ a\cos(u)\sin(v),\ 0\right)$

  $\vec{r}_v = \left(a\cos(u)\cos(v),\ a\sin(u)\cos(v),\ -a\sin(v)\right)$

- $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin(u)\sin(v) & a\cos(u)\sin(v) & 0 \\ a\cos(u)\cos(v) & a\sin(u)\cos(v) & -a\sin(v) \end{vmatrix}$$

  $$= \vec{i}\cdot\left(-a^2\cos(u)\sin^2(v)\right)$$
  $$- \vec{j}\cdot\left(a^2\sin(u)\sin^2(v)\right)$$
  $$+ \vec{k}\cdot\left(-a^2\sin^2(u)\sin(v)\cos(v) - a^2\cos^2(u)\sin(v)\cos(v)\right)$$

$$= \left(-a^2\cos(u)\sin^2(v),\ a^2\sin(u)\sin^2(v),\ -a^2\sin(v)\cos(v)\right)$$

- $$|\vec{r}_u \times \vec{r}_v| = \sqrt{a^4\cos^2(u)\sin^4(v) + a^4\sin^2(u)\sin^4(v) + a^4\sin^2(v)\cos^2(v)}$$

$$= \sqrt{a^4\sin^4(v) + a^4\sin^2(v)\cos^2(v)}$$

$$= a^2\sin(v)$$

- $$SA = \int_0^{2\pi}\int_0^{\pi} a^2\sin(v)\,dv\,du$$

$$= 4\pi a^2$$

Ex: SA of $z = x^2 + y^2$ w/ $x^2 + y^2 \le 1$.

Soln:

- $\vec{r}(u,v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \le 1$

- $\vec{r}_u = (1, 0, 2u)$

  $\vec{r}_v = (0, 1, 2v)$

- $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$$

- $|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + 4u^2 + 4v^2}$

- $$SA = \iint_D \sqrt{1 + 4u^2 + 4v^2}\, dA$$

  $$= \int_0^{2\pi} \int_0^1 r\sqrt{1 + 4r^2}\, dr\, d\theta$$

  $= etc.$

$u = r\cos\theta$

$v = r\sin\theta$.

Rem: When $\vec{r}(u,v) = (u, v, f(u,v))$ for (u,v) in D

SA of graph of f over D = SA of surface $\vec{r}(u,v) = (u, v, f(u,v))$

Surface Integrals

Notn:

- S = surface w/ param. $\vec{r}(u,v)$ w/ (u,v) in D.
- $f(x,y,z)$ = fcn on $\mathbb{R}^3$.
- Divide D up into rectangles R_{ij}
- Spse Area$(R_{ij}) = \Delta u \cdot \Delta v$
- Sample (u_i^*, v_j^*).

Defn:

$$\iint_S f \, dS = \lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^n f(\vec{r}(u_i^*, v_j^*)) \cdot \text{SA of } \vec{r}(R_{ij})$$

$$= \lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^n f(\vec{r}(u_i^*, v_j^*)) \cdot |\vec{r}_u \times \vec{r}_v| \cdot \Delta u \cdot \Delta v$$

$$= \iint_D f(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v| \, dA.$$

Example: $\iint_S x^2\, dS$ where $S =$ unit sphere

Soln:

- $\vec{r}(u,v) = \left(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \right)$
- $|\vec{r}_u \times \vec{r}_v| = \sin(v)$ (from before)
- $\iint_S x^2\, dS = \int_0^{2\pi}\int_0^{\pi} \cos^2(u)\sin^3(v)\, dv\, du$

$$= \int_0^{2\pi} \cos^2(u)\, du \cdot \int_0^{\pi} \sin^3(v)\, dv$$

$$= \text{etc.}$$

Example: $\iint_S z \, dS$ where $S = \{ y = x + z^2, \ 0 \le x \le 1, \ 0 \le z \le 2 \}$

Soln:

- $\vec{r}(u,v) = (u, u + v^2, v)$, $0 \le u \le 1$, $0 \le v \le 2$
- $\vec{r}_u = (1, 1, 0)$

  $\vec{r}_v = (0, 2v, 1)$
- $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & 1 & 0 \\ 0 & 2v & 1 \end{vmatrix} = (1, -1, 2v)$
- $|\vec{r}_u \times \vec{r}_v| = \sqrt{2 + 4v^2}$
- $\iint_S z \, dS = \int_0^1 \int_0^2 v \sqrt{2 + 4v^2} \, dv \, du$

  $= \int_0^1 \int_2^{18} \frac{1}{8} \sqrt{a} \, da \, du$

  $= \frac{1}{8} \cdot \frac{2}{3} \left(18^{3/2} - 2^{3/2} \right)$

$a = 2 + 4v^2$

$da = 8v \, dv$

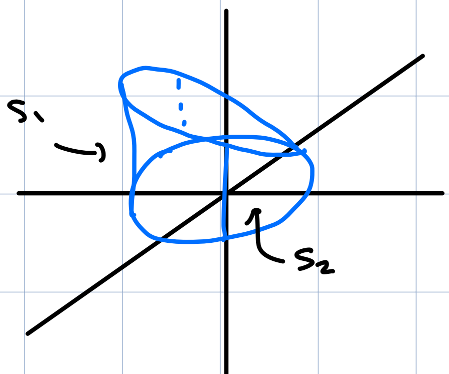
Defn: If S is a union of piecewise smooth surfaces $S_1, \ldots, S_n$, then

$$\iint_S f \, dS = \sum_{i=1}^{n} \iint_{S_i} f \cdot dS$$

Example: $\iint_S z \, dS$ where S is part of surface w/ sides the cylinder $\underline{x^2+y^2=1}$ w/ bottom $\underline{\{z=0,\ x^2+y^2\le 1\}}$ that is bounded above by $z = 1+x$

① ②

Soln: • Draw:



• $\vec{r}_1(u,v) = \left(\cos(u),\ \sin(u),\ v\right)$, $0 \le u \le 2\pi$, $1+\cos(u) \ge v$

$\vec{r}_2(u,v) = \left(v\cos(u),\ v\sin(u),\ 0\right)$, $0 \le u \le 2\pi$, $0 \le v \le 1$

- $\vec{r}_{1u} = (-\sin(u), \cos(u), 0)$

$\vec{r}_{1v} = (0, 0, 1)$

$$\vec{r}_{1u} \times \vec{r}_{1v} = \begin{vmatrix} i & \vec{j} & \vec{k} \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\cos(u), \sin(u), 0)$$

$\Rightarrow |\vec{r}_{u1} \times \vec{r}_{1v}| = 1$

- $\iint_{S_1} z \, dS$

$$= \int_0^{2\pi} \int_0^{1+\cos(u)} v \, dv \, du$$

$$= \int_0^{2\pi} \frac{1}{2} (1 + \cos(u))^2 \, du$$

$$= \int_0^{2\pi} \frac{1}{2} (1 + 2\cos(u) + \cos^2(u)) \, du$$

$$= \pi + \frac{1}{2} \int_0^{2\pi} \cos^2 u \, du$$

$$= \pi + \int_0^{2\pi} \frac{1}{4}\left(1 + \cos(2u)\right)\, du$$

$$= \pi + \frac{\pi}{2} + 0$$

$$= \frac{3\pi}{2}$$

- $\iint_{S_2} z \, dS = \iint_D 0 \, |\vec{r}_u \times \vec{r}_v| \, du\,dv = 0$
- $\Rightarrow \iint_S z \, dS = 3\pi/2$.

Orientations

Defn: A surface S is orientable if one can choose a normal vector at every point in S st the normal vectors vary continuously over S.

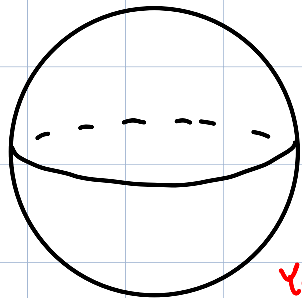
↳ If S is <u>not</u> orientable it is said to be non-orientable

↳ An orientable surface has a notion of up/down

↳ A choice of normal vectors is called an orientation

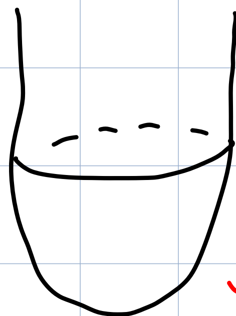
Ex:

1



Yes

2



Yes

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yes

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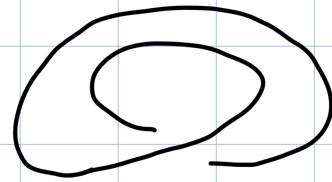

no

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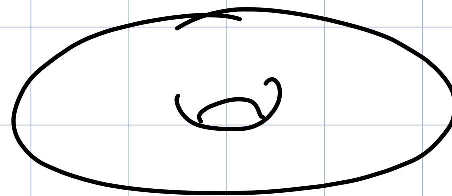

yes

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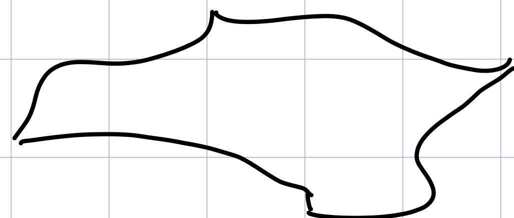
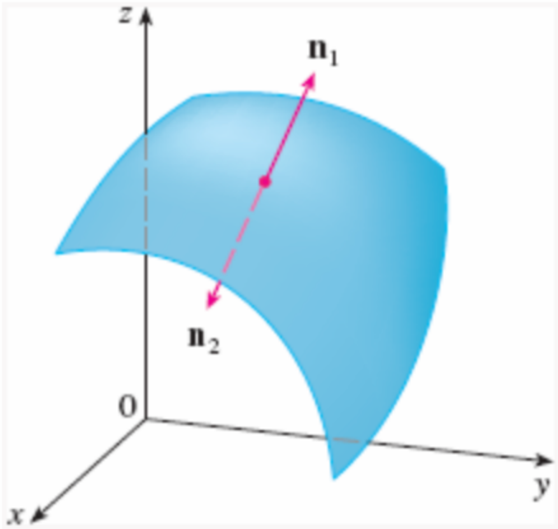


yes

Picture :



The two orientations of an orientable surface