Lecture # 16

Title: Parametric Surfaces

Section: Stewart 16.6

Warmup: Compute the divergence and curl of

$$F(x,y,z) = \left(x^2 + e^z x,\ \sin(xy),\ e^z x - z \right)$$

Soln:

- $$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + e^z x & \sin(xy) & e^z x - z \end{vmatrix}$$

$$= \vec{i}\,(0 - 0) - \vec{j}\,(e^z - x e^z) + \vec{k}\,(y\cos(xy))$$

- $$\nabla \cdot F = 2x + e^z + x\cos(xy) + x e^z - 1$$

Defn: A parametric curve is a map

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

for $a \le t \le b$

↳ Determined by 1-parameter t.

Defn: A parametric surface is a map

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

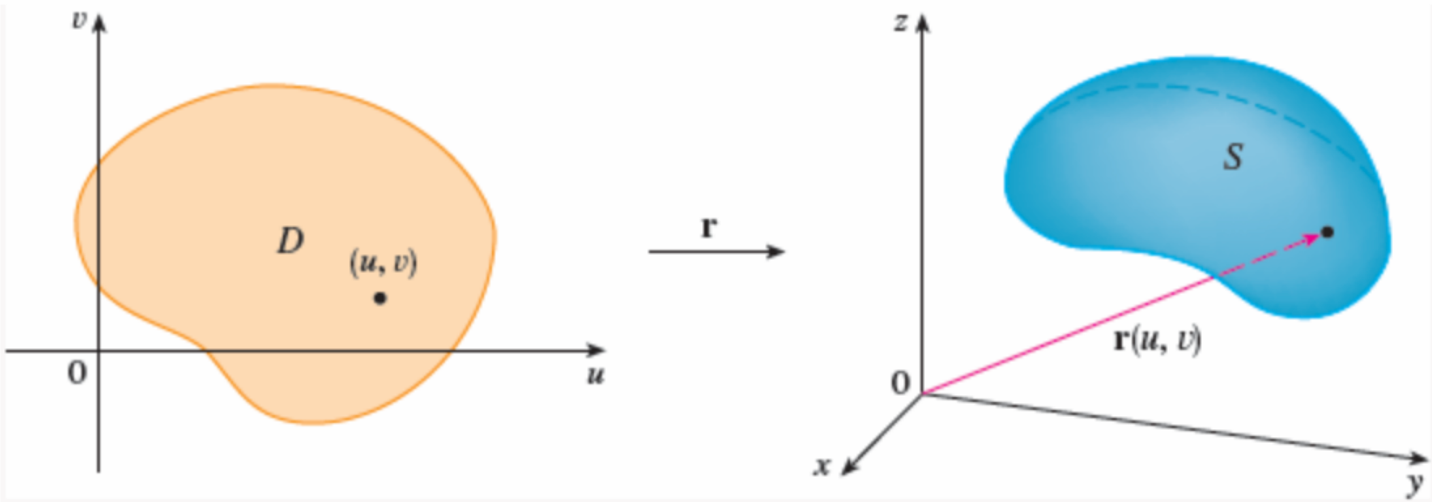
$$= (x(u,v),\ y(u,v),\ z(u,v))$$

So $\vec{r}$ maps some domain D in uv-plane into $\mathbb{R}^3$.

The associated surface, S, is the image of $\vec{r}$ in $\mathbb{R}^3$

↳ A surface can have multiple param.

Picture :



Example: $\vec{r}(u,v) = (2\cos(u), v, 2\sin(u))$

What is the associated surface?

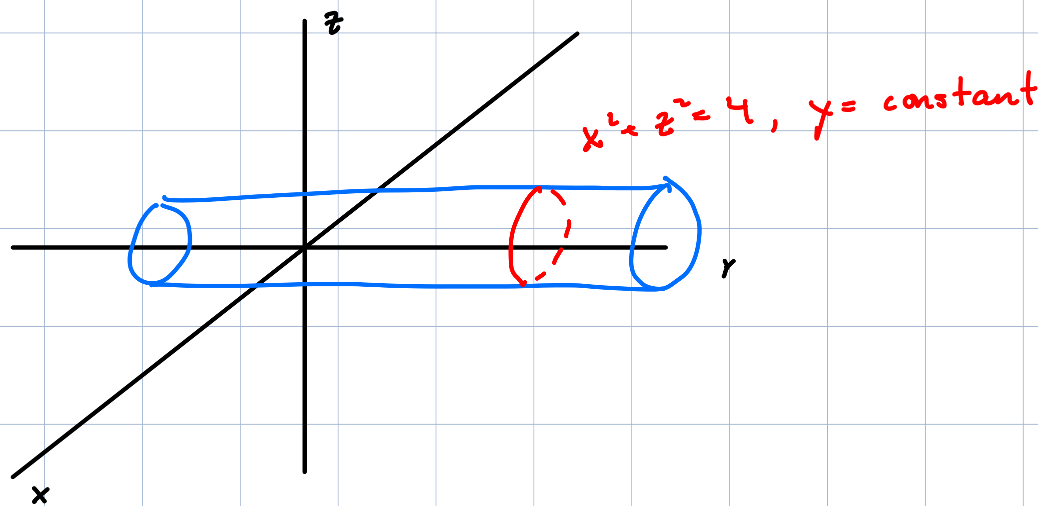
Soln:

- Understand image when u or v is fixed.
- Spse $v = v_0$ is fixed, then as u varies, we get a curve w/ param. u.
- $x = 2\cos(u)$, $y = v_0 =$ "constant", $z = 2\sin(u)$

$\Rightarrow$ $x^2 + z^2 = 4$, $y = v_0$

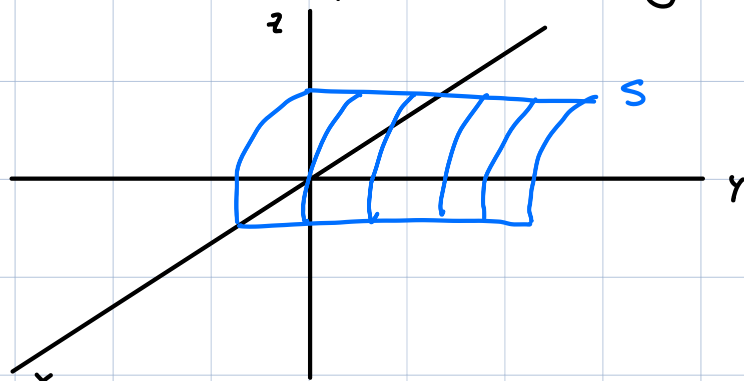
$\Rightarrow$ As u varies, we get circle of radius 2 lifted v_0 amount away from xz-plane

- As v varies we just get family of circles that make up a cylinder!


z
x
y
x² + z² = 4 , y = constant

Ex: Spse $\vec{r}(u,v)$ is as above, but w/ $0 \le u \le \pi/2$, $0 \le v \le 3$.

Soln:

- As $0 \le u \le \pi/2$ => just get first 1/4 of circle.
- As $0 \le v \le 3$ => cylinder has height 3.
- 

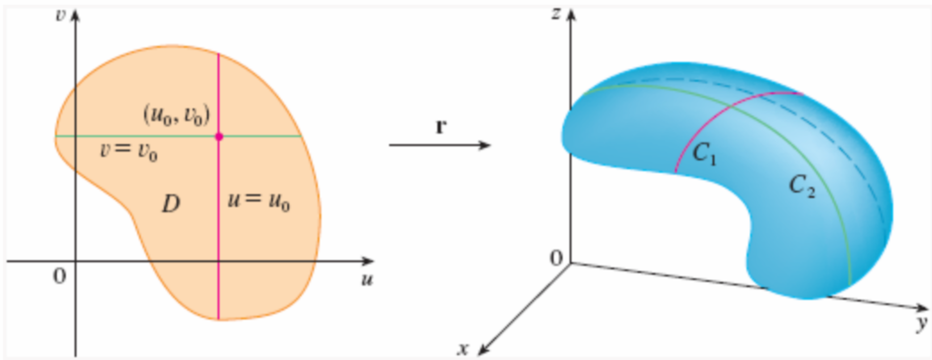
Rem: • To visualize, often helpful to look at curves

$\vec{r}(u_0, v)$ or $\vec{r}(u, v_0)$

* view as "constant / fixed"

and see how they assemble into a surface

Picture:



Defn: $\vec{r}(u_0, v)$ = vertical grid curves

$\vec{r}(u, v_0)$ = horizontal grid curves

Ex 8 $\quad \vec{r}(u,v) = \big((2+\sin(v))\cos(u),\ (2+\sin(v))\sin(u),\ u+\cos(v)\big)$

Describe the associated surface

Soln:

- Fix $v = v_0 = \text{constant}$

  $\vec{r}(u,v_0) = \big((2+\sin(v_0))\cos(u),\ (2+\sin(v_0))\sin(u),\ u+\cos(v_0)\big)$

  $\Rightarrow \quad x^2+y^2 = (2+\sin(v_0))^2$

  $\qquad z = u + \cos(v_0)$

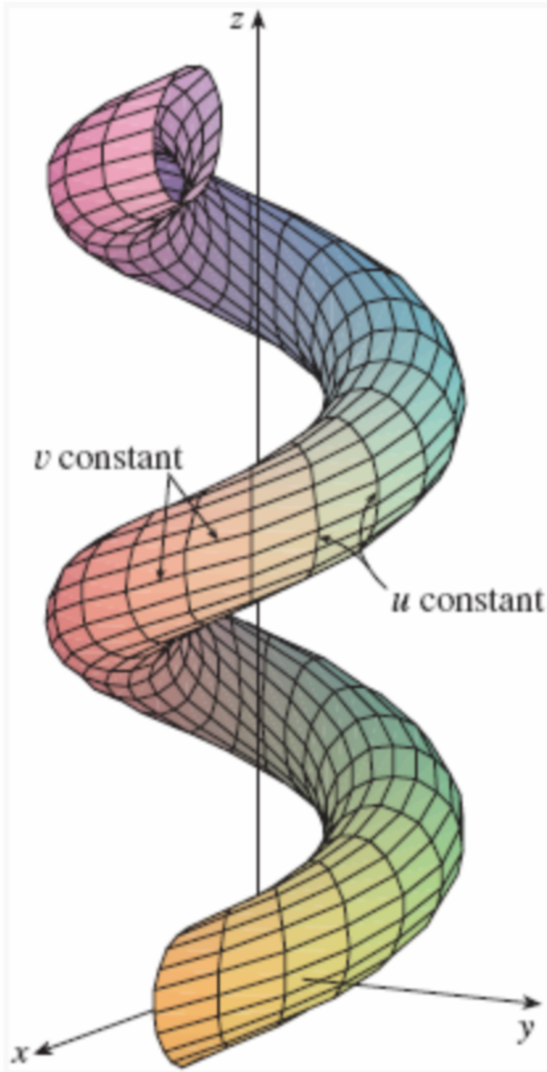
  $\Rightarrow$ hor. grid lines are helixes

- Fix $u = u_0 = \text{constant}$

$$\Rightarrow \left.\begin{aligned} x &= \text{constant} + \text{constant}\cdot\sin(v) \\ y &= \quad \text{"} \qquad - \qquad \text{"} \quad \cdot \sin(v) \\ z &= \quad \text{"} \qquad + \ \cos(v) \end{aligned}\right\}$$

Something periodic in $v \Rightarrow$ circle/ellipse like.

Picture :



Ex: Find parametric eqn for plane that passes through $\vec{r}_0$ and contains vectors $\vec{a}$, $\vec{b}$ (non-parallel).

Soln:

- If $\vec{r}_0 = \vec{0} \Rightarrow \vec{r}(u,v) = u\vec{a} + v\vec{b}$
- If $\vec{r}_0 \neq \vec{0}$, we just shift our surface by $\vec{r}_0$.

$$\Rightarrow \vec{r}(u,v) = \vec{r}_0 + u \cdot \vec{a} + v \cdot \vec{b}.$$

Rem: If $\vec{r} = (x,y,z)$, $\vec{r}_0 = (x_0, y_0, z_0)$

$\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

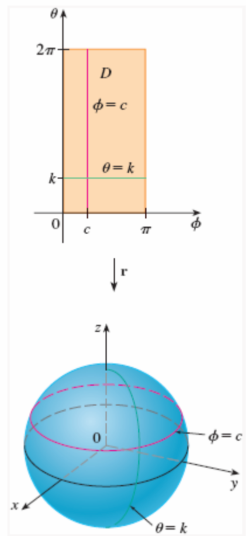
$$\Rightarrow \vec{r}(u,v) = (x_0 + ua_1 + vb_1,\ y_0 + ua_2 + b_2 v,\ z_0 + ua_3 + vb_3)$$

Ex: Find param. eqn for sphere $x^2 + y^2 + z^2 = a^2$

Soln:
- Use spherical coordinates!
- $\vec{r}(u,v) = (a\sin(u)\cos(v),\ a\sin(u)\sin(v),\ a\cos(u))$

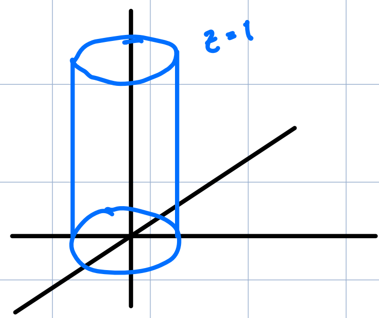
$\overset{\phi}{0 \le u \le \pi}, \quad \overset{\theta}{0 \le v \le 2\pi}$

Picture:



Ex: Find param eqn for cylinder $x^2+y^2=4$, $0 \le z \le 1$

Soln: • Draw:



• $\vec{r}(u,v) = (2\cos(u), 2\sin(u), v)$

$0 \le u \le 2\pi$, $0 \le v \le 1$.

Soln 2: • $\vec{r}(u,v) = (2\cos(u^2), 2\sin(u^2), \sin(v))$

$0 \le u \le \sqrt{2\pi}$, $0 \le v \le \pi/2$

Ex: Find param eqn for $z = x^2 + 2y^2$

Soln: $\vec{r}(u,v) = (u, v, u^2 + 2v^2)$

Ex: Find param eqn for $z = f(x,y)$

Soln: $\vec{r}(u,v) = (u, v, f(u,v))$

Ex: Find param eqn for cone $z = \sqrt{x^2+y^2}$

Soln 1: $\vec{r}(u,v) = (u, v, \sqrt{u^2+v^2})$

Soln 2: $\vec{r}(u,v) = (u\cos(v), u\sin(v), u)$

Tangent Planes

Recall:

Given $\vec{r}(t) = (x(t), y(t))$

$\vec{r}\,'(t) = (x'(t), y'(t))$ is tangent to $\vec{r}(t)$.

$\Rightarrow$ $(x(t_0) + t \cdot x'(t_0), \; y(t_0) + t\, y'(t_0))$

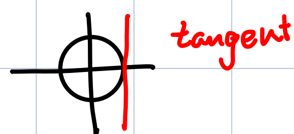
defines tangent line to $\vec{r}(t)$ at t_0.

Ex:

$\vec{r}(t) = (\cos(t), \sin(t))$

$\vec{r}\,'(t) = (-\sin(t), \cos(t))$

$\Rightarrow$ $(1, t)$ is tangent line at $\vec{r}(0) = (1, 0)$.



Notn: Consider $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

Rem: w/ $u = u_0$ fixed, look at tangent vec. to $\vec{r}(u_0, v)$

at (u_0, v_0):

$$\vec{r}_v(t) = \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right)$$

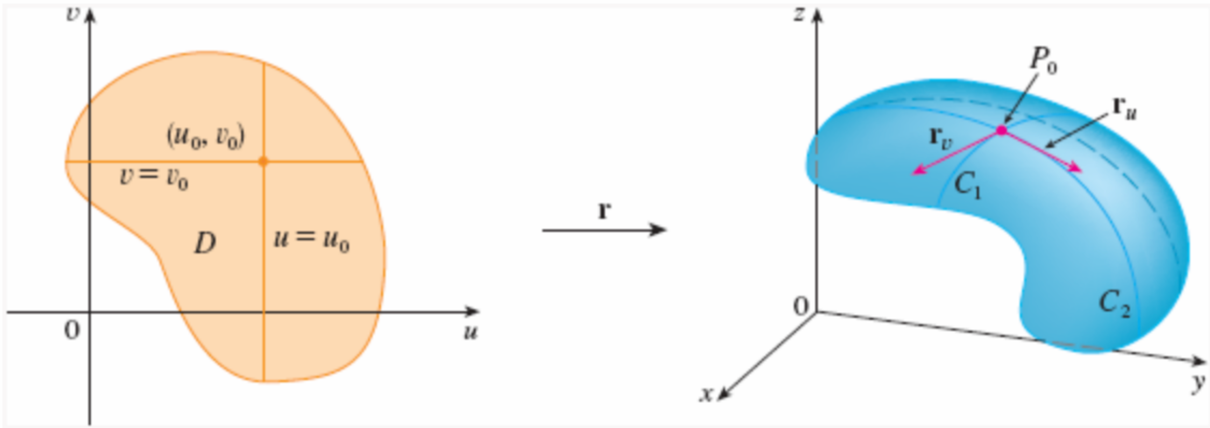
" $v = v_0$ " , " . " " " $\vec{r}(u, v_0)$

" " "

$$\vec{r}_u(t) = \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right)$$

Rem: $\vec{r}_v$, $\vec{r}_u$ define two tangent vectors that det. tangent plane to surface

Picture:



Defn: If $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, then the surface S is called smooth (it has no "corners")

Defn:
- The tangent plane of a smooth surface S is the plane that contains the vectors $\vec{r}_u$, $\vec{r}_v$
- The normal vector to the tangent plane is the vector $\vec{r}_u \times \vec{r}_v$.

Rem: A plane is determined by its ① normal vector and a ② base point.

- $\vec{n} = (a, b, c)$ = normal vector
- $P = (x_0, y_0, z_0)$ = base point

$$\text{Plane} = \{ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \}$$

Question: Why is $\vec{n}$ normal to the plane?

↳ $\vec{n} = \nabla f$ where $f = a(x-x_0) + b(y-y_0) + c(z-z_0)$

↳ $\text{Plane} = f^{-1}(0)$

↳ ∇f is always orthogonal to level sets.

$\Rightarrow \vec{n} \perp$ plane.

Ex:

Consider $\vec{r}(u,v) = (u^2, v^2, u+2v)$ at $(1,1,3)$

What is the normal vector and tangent plane?

Soln:

- $\vec{r}_u = (2u, 0, 1)$

  $\vec{r}_v = (0, 2v, 2)$

- $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = (-2v, -4u, 4uv)$$

- Normal at $(1,1,3)$ is: $(-2,-4,4)$

- Tangent plane at $(1,1,3)$ is:

$$-2(x-1) - 4(y-1) + 4(z-3) = 0$$

Example: $\vec{r}(u,v) = (\sin(u)\cos(v),\ \sin(u)\sin(v),\ \cos(u))$

What is tangent plane at $(1,0,0)$?

Soln:

- $\vec{r}(\pi/2, 0) = (1,0,0)$
- $\vec{r}_u = (\cos(u)\sin(v),\ \cos(u)\sin(v),\ -\sin(u))$

  $\vec{r}_v = (-\sin(u)\sin(v),\ \sin(u)\cos(v),\ 0)$
- $\vec{r}_u(\pi/2, 0) = (0, 0, -1)$

  $\vec{r}_v(\pi/2, 0) = (0, 1, 0)$
- $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = (1, 0, 0)$$
- Plane: $(x-1) = 0$