Lecture # 15

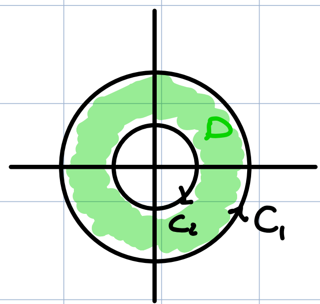
Title: Curl and Divergence (and a little more Green's Theorem)

Section: Stewart 16.5

Warm-up: Compute $\int_C y^2\,dx + 3xy\,dy$ where C is the boundary of the annulus D contained between $r = 1$ and $r = 2$.

Soln:

- Draw:



- Green's thm:

$$\iint_D 3y - 2y \, dA = \int_{C_1} y^2dx + 3xy\,dy + \int_{C_2} y^2dx + 3xy\,dy$$
$$= \int_C y^2dx + 3xy\,dy.$$

- $D = \{ 1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi \}$

- line integral $= \int_0^{2\pi}\int_1^2 r^2 \sin(\theta)\, dr\, d\theta = 0$

Example: Spse Area(D) = 6 and C is the pos. oriented boundary of D. What is $\int_C F \cdot d\vec{r}$ where

$F = (x^2 + y,\ 3x - y^2)$.

Soln:

- $Q_x - P_y = 3 - 1 = 2$
- Green's thm: $12 = \iint_D 2\, dA = \int_C F\, d\vec{r}$

Stokes-like theorems in Dimension 1

$$\text{Functions} \xrightarrow{\frac{d}{dx}} \text{Functions}$$

$$f(b) - f(a) = \int_a^b \frac{d}{dx}(f)\, dx$$

Rem: Jacobian: $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial}{\partial x}(Q) - \frac{\partial}{\partial y}(P) = Q_x - P_y$

So we can state Green's Theorem as follows:

$$\iint_D \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{vmatrix} dA = \int_{\partial D} F \cdot d\vec{r} = \int_{\partial D} P\,dx + Q\,dy$$

where $F = P\vec{i} + Q\vec{j}$.

Stokes-like theorems in Dimension 2

$$\text{Functions} \xrightarrow{\nabla} \text{Vector Fields} \xrightarrow{\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \cdot & \cdot \end{vmatrix}} \text{Functions}$$

$$\underset{f(\vec{r}(b)) - f(\vec{r}(a))}{\overset{f\big|_{\partial C}}{\shortparallel}} \quad \overset{\text{FTLI}}{=\!=} \quad \int_C \nabla f \cdot d\vec{r}$$

$$\int_{\partial D} F \cdot d\vec{r} \quad \overset{\text{Green's Theorem}}{=\!=} \quad \iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dA$$

- $f \longmapsto \nabla f \longmapsto \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ f_x & f_y \end{vmatrix} = 0$

  ↳ Fcn → Vf → Fcn w/ above maps is zero

- If $\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{vmatrix} = 0$, then $F = \nabla f$ for some f

  ↳ If second map takes vf to zero, then vf is conservative

Curl and Divergence

Defn: Let $F = P\vec{i} + Q\vec{j} + R\vec{k}$ be a v.f. on $\mathbb{R}^3$.

The curl of F is the v.f. on $\mathbb{R}^3$

$$\text{Curl}(F) = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$$

Notn: • Let $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ be a vector differential operator.

$$\hookrightarrow \nabla f = \frac{\partial}{\partial x}(f)\vec{i} + \frac{\partial}{\partial y}(f)\vec{j} + \frac{\partial}{\partial z}(f)\vec{k}$$

$$= f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$$

$\leadsto$ Think of ∇ as "vector" w/

$$\vec{i}\text{-component} = \frac{\partial}{\partial x}$$

$$\vec{j} \quad \text{"} \quad = \frac{\partial}{\partial y}$$

$$\vec{k} \quad \text{"} \quad = \frac{\partial}{\partial z}$$

Rem:

- Think of curl in terms of a cross product
- $\text{Curl}(F) = \nabla \times F$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix}$$

$$= (R_y - Q_z)\,\vec{i} + (P_z - R_x)\,\vec{j} + (Q_x - P_y)\,\vec{k}$$

Ex: Compute $\nabla \times F$ where $F = (xz, xyz, -y^2)$

Soln:

- $$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \vec{i} \cdot (-2y - xy) - \vec{j} \cdot (0 - x) + \vec{k} \cdot (yz - 0)$$

$$= (-2y - xy)\,\vec{i} + x\,\vec{j} + yz\,\vec{k}$$

Ex:

Compute Curl(F) where $F = \sin(x)\vec{i} + e^{xyz}\vec{j} + zx^2\vec{k}$

Soln:

- $\text{Curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(x) & e^{xyz} & zx^2 \end{vmatrix}$

$= \vec{i}\cdot(0 - xye^{xyz}) - \vec{j}\cdot(2xz) + \vec{k}\cdot(yze^{xyz})$

$= -xye^{xyz}\,\vec{i} - 2xz\,\vec{j} + yze^{xyz}\,\vec{k}.$

Thm: $$\text{Curl}(\nabla f) = \nabla \times \nabla(f) = 0 \;{}^{*}$$

* F and its comp. fcn have cont. part. deriv.

Proof:

$$\nabla \times (\nabla(f)) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \vec{i} \cdot (f_{zy} - f_{yz}) - \vec{j} \cdot (f_{zx} - f_{xz}) + \vec{k}(f_{yx} - f_{xy})$$

mixed partials commute

$$= \vec{0}$$

$\square$

Cor: If F is conservative, then $\nabla \times F = 0$.

Example: Is $F = (e^x, e^y, e^{xy})$ conservative?

Soln: • $\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & e^y & e^{xy} \end{vmatrix}$

$= \vec{i}\,(x e^{xy} - 0) + \text{stuff}$

$\neq 0.$

Thm: (When F is defined everywhere w/ components having continuous partial derivatives)

F is conservative iff $\nabla \times F = 0$

Proof: Uses Stoke's thm, which is an analogue of Green's theorem.

↪ See later.

Ex: Show $F = (y^2z^3, 2xyz^3, 3xy^2z^2)$ is conservative and find f st $\nabla f = F$.

Soln:

- $$\nabla \times F = \begin{vmatrix} i & \vec{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= i\,(6xyz^2 - 6xyz^2)$$
$$- \vec{j}\,(3y^2z^2 - 3y^2z^2)$$
$$+ \vec{k}\,(2yz^3 - 2yz^3)$$

$$= 0$$

$\Rightarrow$ conservative

- $\frac{\partial f}{\partial x} = y^2 z^3$

  $\Rightarrow f(x,y,z) = xy^2z^3 + g(y,z)$ Integrate

- $\frac{\partial f}{\partial y} = 2xyz^3$

  $\Rightarrow 2xyz^3 + \frac{\partial g}{\partial y} = 2xyz^3$ diff.

  $\Rightarrow \frac{\partial g}{\partial y} = 0$

  $\Rightarrow$ g only depends on z

- $\frac{\partial f}{\partial z} = 3xy^2z^2$

  $\Rightarrow 3xy^2z^2 + g'(z) = 3xy^2z^2$ diff

  $\Rightarrow g'(z) = 0$

- $\Rightarrow f(x,y,z) = 3xy^2z^2 + C$

Ex :

Show $F = (\sin(y), x\cos(y) + \cos(z), -y\sin(z))$ is conservative and find f st $\nabla f = F$.

Soln :

- $$\nabla \times F = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(y) & x\cos(y) + \cos(z) & -y\sin(z) \end{vmatrix}$$

$$= i \cdot (-\sin(z) - (-\sin(z)))$$
$$- j \cdot (0 - 0)$$
$$+ \vec{K} \cdot (\cos(y) - \cos(y))$$
$$= \vec{0}$$

$\Rightarrow$ conservative

- $f_x = \sin(y)$

$$\Rightarrow f(x,y,z) = x\sin(y) + g(y,z)$$

$f_y = x\cos(y) + \cos(z)$

$$\Rightarrow \cancel{x\cos(y)} + g_y(y,z) = \cancel{x\cos(y)} + \cos(z)$$

$$\Rightarrow g(y,z) = y\cos(z) + h(z)$$

$f_z = -y\sin(z)$

$$\Rightarrow -\cancel{y\sin(z)} + h'(z) = -\cancel{y\sin(z)}$$

$$\Rightarrow h'(z) = 0$$

$$\Rightarrow f = x\sin(y) + y\cos(z)$$

Divergence

Defn: • Let $F = P\vec{i} + Q\vec{j} + R\vec{k}$ be a v.f. on $\mathbb{R}^3$.

The divergence of F is the function on $\mathbb{R}^3$ given by

$$\operatorname{div}(F) = P_x + Q_y + R_z$$

Rem:

$$\begin{aligned}\operatorname{div}(F) &= \nabla \cdot F \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (P, Q, R) \\ &= P_x + Q_y + R_z\end{aligned}$$

Ex: $F = (xz, xyz, -y^2)$, compute $\mathrm{div}(F)$

Soln: $\mathrm{div}(F) = \nabla \cdot F = z + xz$

Ex: $F = (e^x yz, z^2 xy, \sin(xz))$

Soln: $\nabla \cdot F = e^x yz + z^2 x + x\sin(xz)$

Theorem: $\mathrm{div}(\mathrm{curl}(F)) = \nabla \cdot (\nabla \times F) = 0$

Proof: $\nabla \cdot (\nabla \times (P, Q, R))$

$$= \nabla \cdot (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

$$= \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$= 0$$

$\square$

Ex: $F = (xz, xyz, -y^2)$ is not the curl of a vector field, ie, $F \neq \nabla \times G$ for some vector field G.

$\hookrightarrow$ If so, $z + xz = \nabla \cdot F = \nabla \cdot \nabla \times G = 0$.

a contradiction.

$\square$

Defn: If $\nabla \cdot F = 0$, then F is said to be incompressable

Stokes like theorems in Dimension 3

$$\text{Functions} \xrightarrow{\nabla} \text{Vector fields} \xrightarrow[\text{curl}]{\nabla\times} \text{Vector fields} \xrightarrow[\text{div}]{\nabla\cdot} \text{Functions}$$

$$f\big|_{\partial c} \overset{\text{FTLI}}{=\!=} \int_c \nabla f \cdot d\vec{r}$$

$$\int_? ? \; d? = \int_? \nabla \times ? \; d?$$

$$\int_? ? \; d? = \int_? \nabla \cdot ? \; d?$$

- $\nabla \times (\nabla f) = 0$, $\nabla \cdot (\nabla \times F) = 0$

  ↳ Apply two operators in a row gives 0 !