Lecture # 14

Title: Green's Theorem

Section: Stewart 16.4

Theorem: Fundamental Theorem for line integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

where $\vec{r}(t)$ is defined for $a \le t \le b$.

Warm-up: Consider $F(x,y) = \sin(y)\,\vec{i} + (1 + x\cos(y))\,\vec{j}$.

Compute $\int_C F \cdot d\vec{r}$ where C is the curve given by $\vec{r}(t) = e^t \sin(e^t)\,\vec{i} + t^{1/2}\,\vec{j}$ for $0 \le t \le 1$

Soln:

- Is F conservative?: $P_y = \cos(y) = Q_x \Rightarrow$ yes!
- Compute f st $\nabla f = F$.

$$\frac{\partial f}{\partial x} = P(x,y) = \sin(y)$$

$$\Rightarrow \quad f(x,y) = x\sin(y) + g(y)$$

$$\Rightarrow \quad x\cos(y) + g'(y) = \frac{\partial f}{\partial y} = Q(x,y) = 1 + x\cos(y)$$

$$\Rightarrow \quad g'(y) = 1$$

$$\Rightarrow \quad g(y) = y$$

$$\Rightarrow \quad f(x,y) = x\sin(y) + y$$

- Use Fund. thm of line integrals

$\hookrightarrow \int_c F \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r}$

$= f(\vec{r}(1)) - f(\vec{r}(0))$

$= e\sin(e) \cdot \sin(1) + 1$

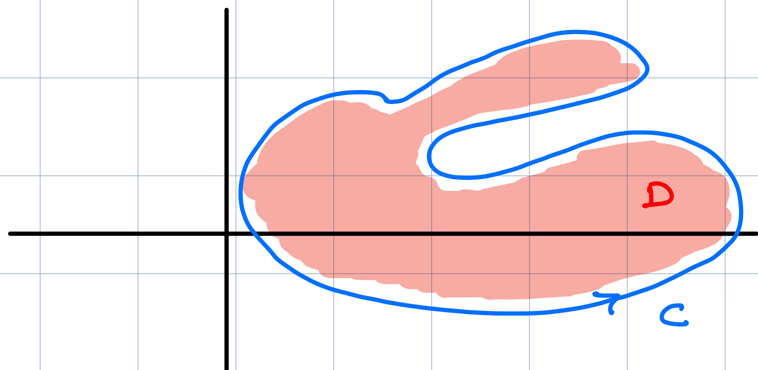
$\vec{r}(1) = e\sin(e)\vec{i} + \vec{j}$

$\vec{r}(0) = \sin(1)\vec{i}$

Notn:

- Let D = region in $\mathbb{R}^2$.
- Let C be the boundary of D oriented/directed st "left of C" points into D (positively oriented)

Picture:



Thm: (Green's Theorem): Spse $F(x,y) = P\,\vec{i} + Q\,\vec{j}$

$$\int_C P\,dx + Q\,dy = \int_C F \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Notn:

- ∂D = boundary of D w/ positive orientation.

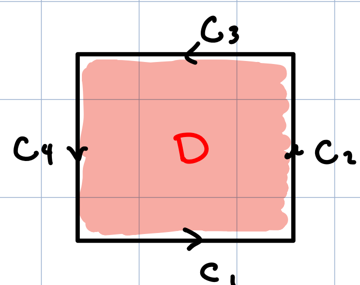
$$\Rightarrow \int_{\partial D} F \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Proof:

- Spse D is a rectangle
- After a change of variables we can assume

$$D = [0,1] \times [0,1]$$

Write



- Spse $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$
- $$\iint_D Q_x - P_y \, dA$$

$$= \int_0^1\int_0^1 Q_x(x,y) - P_y(x,y)\, dx\,dy$$

$$= \int_0^1\int_0^1 Q_x(x,y)\, dx\,dy - \int_0^1\int_0^1 P_y(x,y)\, dy\,dx$$

$$= \int_0^1 Q(1,y) - Q(0,y)\, dy - \int_0^1 P(x,1) - P(x,0)\, dx$$

$$= \int_{C_2} F\cdot d\vec{r} + \int_{C_4} F\cdot d\vec{r} + \int_{C_3} F\cdot d\vec{r} + \int_{C_1} F\cdot d\vec{r}$$

$$= \int_C F\cdot d\vec{r}$$

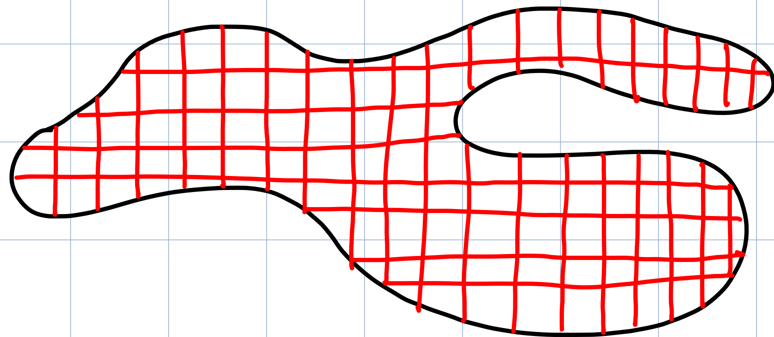
* e.g. $\vec{r}_2(t) = \vec{i} + t\vec{j}$ for $0 \le t \le 1 \Rightarrow \vec{r}_2\,'(t) = \vec{j}$

$$\Rightarrow \int_{C_2} F\cdot d\vec{r} = \int_0^1 F(r_2(t))\cdot \vec{r}_2\,'(t)\, dt$$

$$= \int_0^1 (P(1,t), Q(1,t))\cdot(0,1)\, dt$$

$$= \int_0^1 Q(1,t)\, dt$$

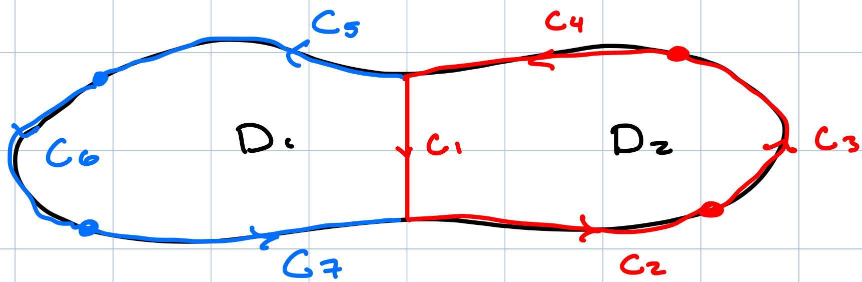
- When D is more general, we divide D up into "rectangles".



and "sum" the above argument over all the "rectangles"

- Some care is required near the boundary
  ↳ Need change of variables and inverse function theorem

- Why can we sum: things cancel!



$$\iint_D P_x - Q_y \, dA$$

$$= \iint_{D_1} P_x - Q_y \, dA + \iint_{D_2} P_x - Q_y \, dA$$

$$= \int_{C_5} F \cdot d\vec{r} + \int_{C_6} F \cdot d\vec{r} + \int_{C_7} F \cdot d\vec{r} - \cancel{\int_{C_1} F \cdot d\vec{r}}$$

$$+ \cancel{\int_{C_1} F \cdot d\vec{r}} + \int_{C_2} F \cdot d\vec{r} + \int_{C_3} F \cdot d\vec{r} + \int_{C_4} F \cdot d\vec{r}$$

$$= \int_{\partial D} F \cdot d\vec{r}$$

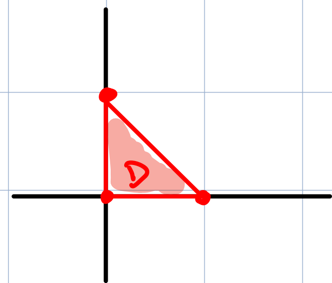
$\square$

Example: Compute $\int_C x^4\,dx + xy\,dy = \int_C (x^4\vec{i} + xy\vec{j}) \cdot d\vec{r}$

where C is boundary of the triangle w/ vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Soln:

- Draw:



- Apply Green's Theorem

$$\hookrightarrow \int_C x^4\,dx + xy\,dy = \iint_D y\,dA$$

$$= \int_0^1 \int_0^{-x+1} y\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 (-x+1)^2\,dx$$

$$= 1/6$$

Example:

Compute $\int_C F \cdot d\vec{r}$ where

$$F(x,y) = (3y - e^{\sin(x)})\,\vec{i} + (7x + \sqrt{y^4+1})\,\vec{j}$$

and C is the circle of radius 3.

Soln:

- Green's Thm: $D = \{ r \leq 3 \}$

$$\hookrightarrow \int_C F \cdot d\vec{r} = \iint_D 7 - 3 \; dA$$

$$= 4 \iint_D dA$$

$$= 4 \pi \cdot (3)^2$$

$$= 36\pi .$$

Cor:

$$\text{Area}(D) = \iint_D 1 \, dA = \int_{\partial D} x \, dy$$
$$= \int_{\partial D} -y \, dx$$
$$= \frac{1}{2} \int_{\partial D} x\,dy - y\,dx$$

Example: Compute area of ellipse $x^2/a^2 + y^2/b^2 = 1$.

Soln:

- $\text{Area}(D) = \int_{\partial D} \frac{1}{2}(x\,dy - y\,dx)$

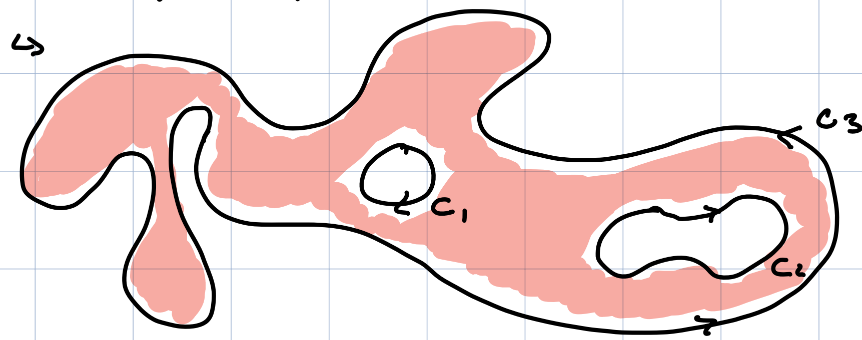
  $\vec{r}(t) = a\cos(t)\,\vec{i} + b\sin(t)\,\vec{j}$ for $0 \le t \le 2\pi$

- $\vec{r}\,'(t) = -a\sin(t)\,\vec{i} + b\cos(t)\,\vec{j}$

- $\text{Area}(D) = \frac{1}{2}\int_0^{2\pi} a\cos(t)\, b\cos(t) - b\sin(t)\cdot(-a\sin(t))\, dt$

$$= \frac{1}{2}\int_0^{2\pi} ab \, dt$$
$$= ab \cdot \pi$$

Rem: D could be more complicated and have several boundary components.



If ∂D is union of finite # of simple closed curves, then

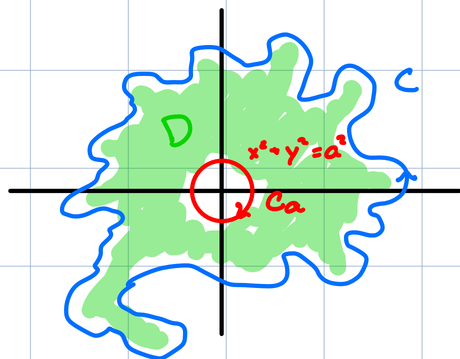
$$\iint_D (Q_x - P_y)\, dA = \sum_{i=1}^{n} \int_{C_i} P\,dx + Q\,dy$$

Example: Consider $F = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$.

Show that $\int_C F \cdot d\vec{r} = 2\pi$ for <u>every</u> closed curve that goes positively about the origin.

Soln:

- Draw:



- Green's Thm $\Rightarrow$

$$\iint_D Q_x - P_y \, dA = \int_C F \cdot d\vec{r} - \int_{-C_a} F \cdot d\vec{r}$$

- $Q_x = \dfrac{y^2 - x^2}{(x^2+y^2)^2} = P_y \Rightarrow \iint_D Q_x - P_y \, dA = 0$

- $\Rightarrow \displaystyle\int_C F \cdot d\vec{r} = \int_{-C_a} F \cdot d\vec{r}$

  $\Rightarrow$ suffices to show that $\int_{-C_a} F \cdot d\vec{r} = 2\pi$

- $\vec{r}(t) = a\cos(t)\,\vec{i} + a\sin(t)\,\vec{j} \qquad 0 \le t \le 2\pi$

  $\vec{r}\,'(t) = -a\sin(t)\,\vec{i} + a\cos(t)\,\vec{j}$

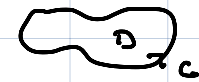
- $\displaystyle\int_{-C_a} F \cdot d\vec{r} = \int_0^{2\pi} \left(\frac{-a\sin(t)}{a^2}, \frac{a\cos(t)}{a^2} \right) \cdot \left(-a\sin(t), a\cos(t)\right) dt$

  $\displaystyle = \int_0^{2\pi} dt$

  $= 2\pi.$

Theorem: (1) $P_y = Q_x$ if and only if (2) F is conservative if and only if (3) $\int_C F \cdot d\vec{r} = 0$ for all closed curves C.

Proof: (1) $\Rightarrow$ (3): Green's thm says



$$0 = \iint_D Q_x - P_y \, dA = \int_C F \cdot d\vec{r}$$

for all closed curves C.

(3) $\Leftrightarrow$ (2): Discussed Last time

(2) $\Rightarrow$ (1): Discussed last time

$\square$

Question: Why does this not contradict the above computation?

Answer: We can only use Green's thm on regions where P_y, Q_x, P, Q, P_x, Q_y are continuous and well defined

↳ Above we have a discontinuity at the origin.