Lecture # 13

Title: The Fundamental Theorem for line integrals

Section: Stewart 16.3

Defn:

Let $F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ be a vector field.

The line integral of F along (directed) C is

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

$$= \int_a^b P(\vec{r}(t)) \cdot x'(t) + Q(\vec{r}(t)) \cdot y'(t) + R(\vec{r}(t))\, z'(t)\, dt$$

$$= \int_a^b F(\vec{r}(t)) \cdot r'(t)\, dt$$

$$=: \int_C F \cdot d\vec{r}$$

Warm-up: Let $F(x,y) = e^x \cdot \vec{i} + x^2 y \cdot \vec{j}$

Let C_1 = straight-line from $(0,0)$ to $(1,0)$.

Let C_2 = graph of $f(x) = x(x-1)$ from $(0,0)$ to $(1,0)$.

Compute $\int_{C_1} F \cdot d\vec{r}$ and $\int_{C_2} F \cdot d\vec{r}$.

Soln:

- $\vec{r}_1(t) = t\,\vec{i} + 0 \cdot \vec{j}$, $0 \leq t \leq 1$

  $\vec{r}_2(t) = t\,\vec{i} + t(t-1)\,\vec{j}$, $0 \leq t \leq 1$

- $\vec{r}_1{}'(t) = \vec{i}$

  $\vec{r}_2{}'(t) = \vec{i} + (2t-1)\,\vec{j}$

- $$\int_{C_1} F \cdot d\vec{r} = \int_0^1 F(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\, dt$$

  $$= \int_0^1 (e^t, t^2 \cdot 0) \cdot (1,0)\, dt$$

$$= \int_0^1 e^t \, dt$$

$$= e - 1$$

- $$\int_{C_2} F \cdot d\vec{r} = \int_0^1 F(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt$$

$$= \int_0^1 \left(e^t,\ t^2 \cdot t(t-1)\right) \cdot (1,\ 2t-1)\, dt$$

$$= \int_0^1 e^t + t^2 \cdot t(t-1)(2t-1)\, dt$$

$$= \int_0^1 e^t + 2t^5 - 3t^4 + t^3\, dt$$

$$= \left(e^t + \frac{t^6}{3} - \frac{3t^5}{5} + \frac{t^4}{4}\right)\Big|_0^1$$

$$= e^t - 1 + 1/3 - 3/5 + 1/4$$

Example:

$$F(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{K}$$

$$\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,z \quad \text{for} \quad 0 \le t \le 1$$

$$\int_C F \cdot d\vec{r} = ?$$

Soln:

$$\int_C F \cdot d\vec{r} = \int_0^1 (t^3, t^5, t^4) \cdot (1, 2t, 3t^2)\, dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6 \, dt$$

$$= 27/28$$

Theorem: Fundamental Theorem for line integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

where $\vec{r}(t)$ is defined for $a \le t \le b$.

Proof:

$$\int_C \nabla f \cdot d\vec{r}$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b f_x(\vec{r}(t)) \cdot x'(t) + f_y(\vec{r}(t)) \cdot y'(t) + f_z(\vec{r}(t))\, z'(t)\, dt$$

chain rule

$$= \int_a^b \frac{d}{dt}\Big(f(\vec{r}(t))\Big)\, dt$$

FTC

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

$\square$

Remark: The above result says: If C goes from (x_0, y_0) to (x_1, y_1) and $F = \nabla f$ is conservative, then

$$\int_C F \cdot d\vec{r} = f(x_1, y_1) - f(x_0, y_0)$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

↪ Similar statement for dimension 3.

Cor: If C_1 and C_2 are piecewise smooth curves w/ the same starting and ending points, then

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot dr$$

Proof:

$$\int_{C_1} \nabla f \cdot d\vec{r} = f(\vec{r}_1(b)) - f(\vec{r}_1(a))$$
$$= f(\vec{r}_2(b)) - f(\vec{r}_2(a))$$
$$= \int_{C_2} \nabla f \cdot dr$$

$\square$

Defn: Given a vector field F. $\int_C F \cdot d\vec{r}$ is path independent if $\int_{C_1} F \cdot d\vec{r} = \int_{C_2} F \cdot d\vec{r}$ for <u>any</u> two paths C_1, C_2 w/ the same starting and ending points

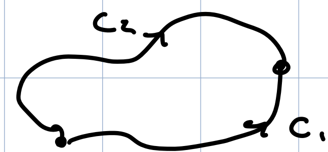
Example: Line integrals of conservative vector fields are independent of paths.

Defn: A curve C is <u>closed</u> if its starting and ending points agree.

Lemma: $\int_C F \cdot d\vec{r} = 0$ for all closed curves C if and only if line integrals of F are path independent.

Proof:

- Spse we have paths C_1, C_2



- Let C be the curve going via C_1 then via $-C_2$
- $\int_C F \cdot d\vec{r} = \int_{C_1} F \cdot d\vec{r} + \int_{-C_2} F \cdot d\vec{r} = \int_{C_1} F \cdot d\vec{r} - \int_{C_2} F \cdot d\vec{r}$
- $\Rightarrow \int_C F \, d\vec{r} = 0$ if and only if $\int_{C_1} F \cdot d\vec{r} = \int_{C_2} F \cdot d\vec{r}$ $\square$

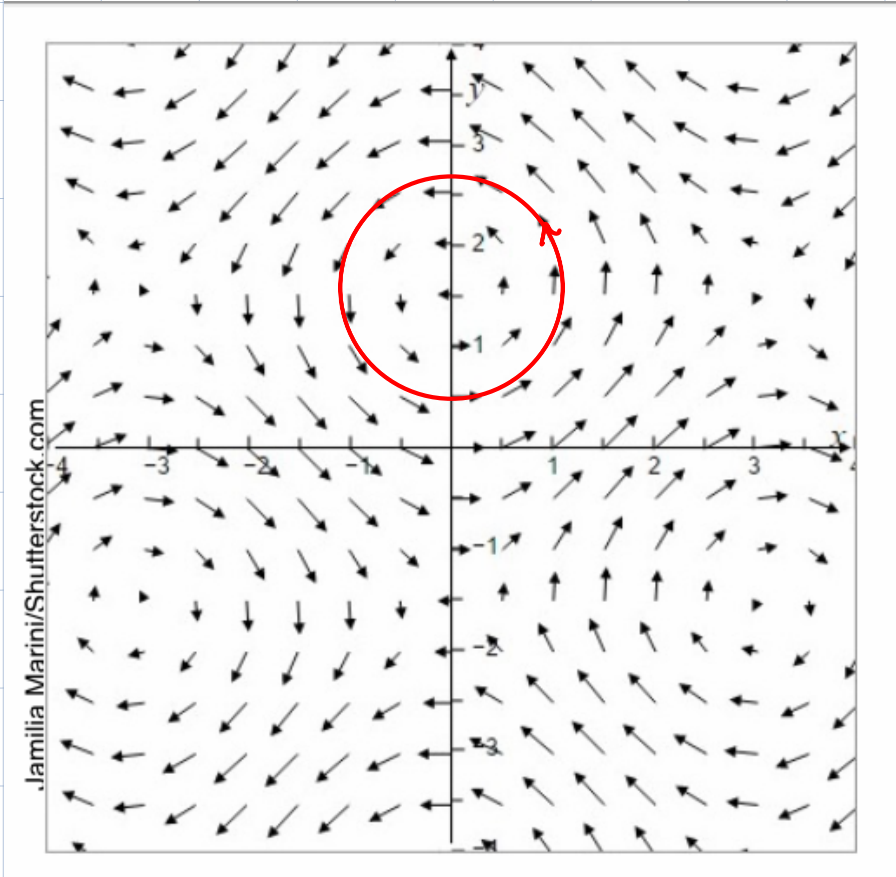
Theorem: $\int_C F \cdot d\vec{r}$ is path independent if and only if F is a conservative vector field ($\nabla f = F$).

Proof: Too complicated for this course.

Remark: If F is conservative, then the total amount that any closed curve C "goes w/" or "goes against" the direction of F is zero.

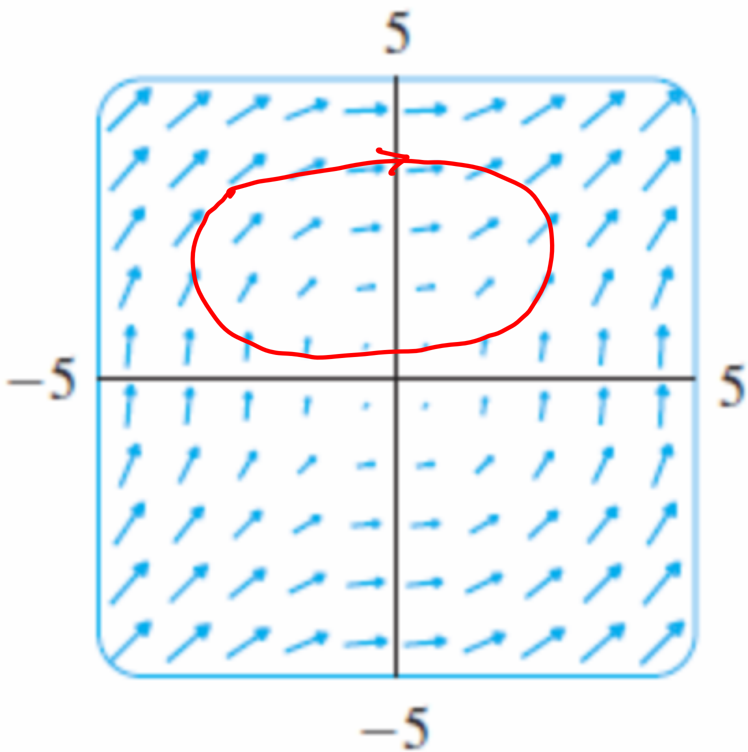
↳ This is the sense in which they are conservative.

Example: Is the following vector field conservative?



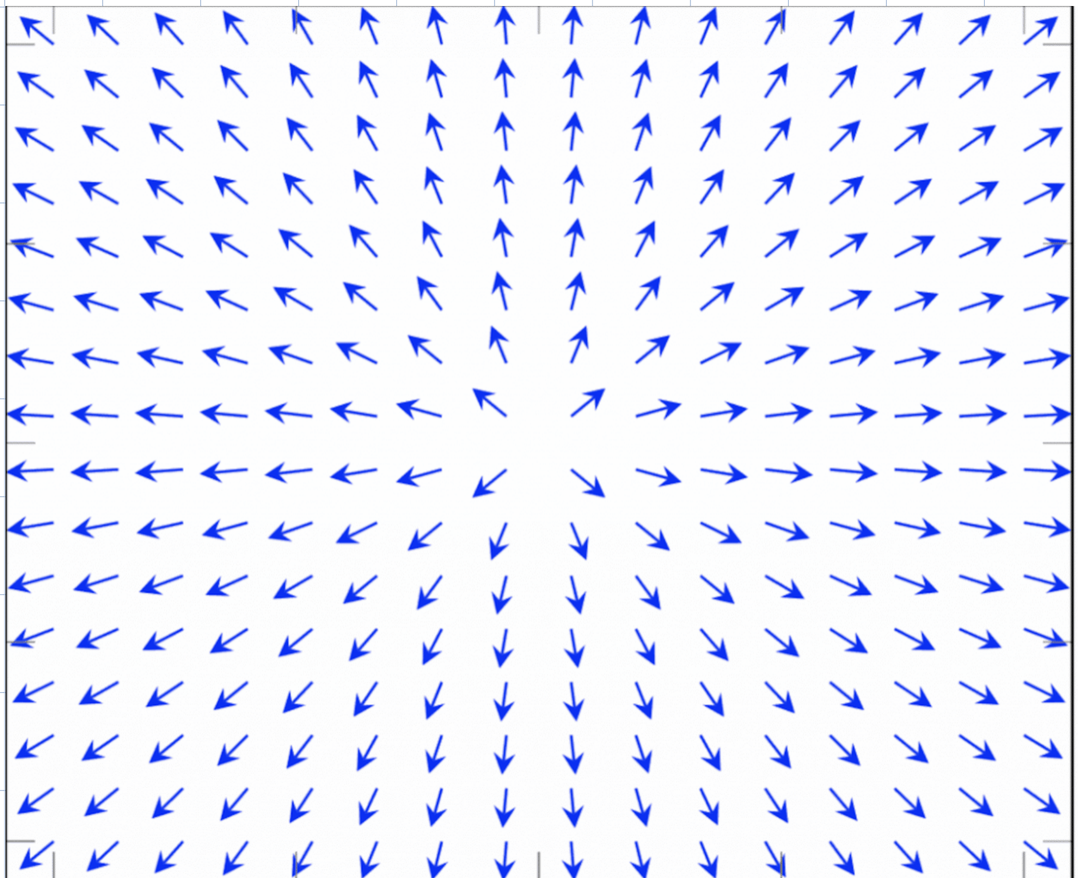
No

Example: Is the following vector field conservative?



I don't think so

Example: Is the following vector field conservative?



Yes...?

Question: How can we determine whether or not a vector field F is conservative?

$\hookrightarrow$ Let's just think about dim 2.

Remark: Spse $F = P\vec{i} + Q\vec{j}$ is conservative.

$\Rightarrow \nabla f = F$ for some fcn f

$\Rightarrow f_x = P, \; f_y = Q$

$$\Rightarrow \frac{\partial F}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial x}$$

Thm: If $F = P\vec{i} + Q\vec{j}$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Thm: (Uses Green's Theorem) $F = P\vec{i} + Q\vec{j}$ is conservative if and only if $P_y = Q_x$.

Example: Which of the following vector fields are conservative.

① $F(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$

② $F(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$

③ $F(x,y) = -y\vec{i} + x\vec{j}$

Soln:

① $P_y = -1$, $Q_x = 1$ $\Rightarrow$ not conservative

② $P_y = 2x$, $Q_x = 2x$ $\Rightarrow$ conservative

③ $P_y = -1$, $Q_x = 1$ $\Rightarrow$ not conservative

Question: If we know F is conservative, how do we find f st $\nabla f = F$.

Example: $F(x,y) = (3+2xy)\,\mathbf{i} + (x^2-3y^2)\,\mathbf{j}$.

Find f st $\nabla f = F$.

Soln:

- $\frac{\partial f}{\partial x} = P(x,y) = 3 + 2xy$

$\Rightarrow \int 3+2xy \, dx = f + g(y)$

$\Rightarrow 3x + x^2 y + g(y) = f(x,y)$

- $\frac{\partial f}{\partial y} = Q(x,y) = x^2 - 3y^2$

$\Rightarrow x^2 + g'(y) = x^2 - 3y^2$

$\Rightarrow g'(y) = -3y^2$

$$\Rightarrow \quad g(y) = -y^3 + C$$

- $f(x,y) = 3x + x^2 y - y^3 + C$

Example: $\int_C F \cdot d\vec{r}$ w/ F as above and C given by

$$\vec{r}(t) = e^t \cos(t)\,\vec{i} + e^t \sin(t)\,\vec{j}$$

for $0 \leq t \leq 2\pi$

Soln: $$\int_C F \cdot d\vec{r} \overset{\text{by above}}{=} \int_C \nabla f \cdot d\vec{r}$$

$$= f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(e^{2\pi}, 0) - f(1,0)$$

$$= 3e^{2\pi} - 3$$

$$= 3(e^{2\pi} - 1)$$

Example:

$F(x,y) = (1+xy)e^{xy}\,\vec{i} + x^2 e^{xy}\,\vec{j}$

① Show that F is conservative and ② find f st $\nabla f = F$.

Use this to ③ compute $\int_C F \cdot d\vec{r}$ where

$$\vec{r}(t) = \cos(t)\,\vec{i} + 2\sin(t)\,\vec{j} \qquad (0 \le t \le \pi/2)$$

Soln:

- $$\left.\begin{aligned} P_y &= xe^{xy} + (1+xy)xe^{xy} \\ Q_x &= 2xe^{xy} + x^2ye^{xy} \end{aligned}\right\} \Rightarrow \text{conservative}$$

- $\dfrac{\partial f}{\partial y} = Q(x,y)$

$$\Rightarrow f(x,y) = \int x^2 e^{xy}\,dy + g(x)$$

$$= xe^{xy} + g(x)$$

- $\dfrac{\partial f}{\partial x} = P(x,y)$

$$\Rightarrow (1+xy)e^{xy} = 1\cdot e^{xy} + xye^{xy} + g'(x)$$

$\Rightarrow \quad g'(x) = 0$

- $\Rightarrow \quad f(x,y) = xe^{xy}$
- $\int_C F \cdot d\vec{r} = \int_C \nabla(xe^{xy}) \cdot d\vec{r}$

$$= \cos(\pi/2)\exp\left(2\cos(\pi/2)\cdot\sin(\pi/2)\right)$$

$$- \cos(0)\exp\left(2\cos(0)\cdot\sin(0)\right)$$

$$= -1$$