Lecture # 12

Title: Line integrals

Section: Stewart 16.2

Defn: A vector field on $\mathbb{R}^3$ is a fcn F that assigns to each point (x, y, z) a vector in $\mathbb{R}^3$.

$$F(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\cdot\vec{k}$$

Notn:

- Let $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$ be a parametric curve in $\mathbb{R}^3$ for some curve C and $a \le t \le b$.
- $\vec{r}\,'(t) = x'\,\vec{i} + y'\,\vec{j} + z'\,\vec{k}$ is the velocity of r.

Defn:

- $\int_C f(x,y,z)\,ds$

$$= \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

- This is the line integral of f along.
- This only depended on C, not the parameterization $\vec{r}$.

Example: Compute $\int_C x^2 + y \, ds$ where C is the straight line from $(1,2)$ to $(5,4)$

Soln:

- $\vec{r}(t) = (1-t)\cdot(i + 2\vec{j}) + t\cdot(5\vec{i} + 4\vec{j})$

  $= (1+4t)\vec{i} + (2 + 2t\vec{j})$

- $\vec{r}\,'(t) = 4\vec{i} + 2\vec{j}$

- $|\vec{r}\,'(t)| = \sqrt{16 + 4} = \sqrt{20}$

- $\int_C x^2 + y \, ds = \int_0^1 ((1+4t)^2 + 2 + 2t)\sqrt{20}\, dt$

  $= \int_0^1 (16t^2 + 8t + 1 + 2 + 2t)\sqrt{20}\, dt$

  $= \sqrt{20}\left(\frac{16}{3}t^3 + 4t^2 + t^2 + 3t\right)\Big|_0^1$

  $= \sqrt{20}\left(\frac{16}{3} + 8\right)$

Notn: • C = curve w/ direction and $\vec{r}(t)$ follows this direction

Defn: • $$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*)) \cdot \Delta x_i$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(r(t_i^*)) \cdot x'(t_i^*) \cdot \Delta t_i$$
$$= \int_C f(x(t), y(t)) \cdot x'(t)\,dt$$

$$\Delta x_i = x(t_i) - x(t_{i-1}), \quad \Delta t_i = t_i - t_{i-1}$$

This is the line integral of f along C wrt x

• $$\int_C f(x,y)\,dy = \int_C f(x(t), y(t)) \cdot y'(t)\,dt$$

This is the line integral of f along C wrt y

Example:

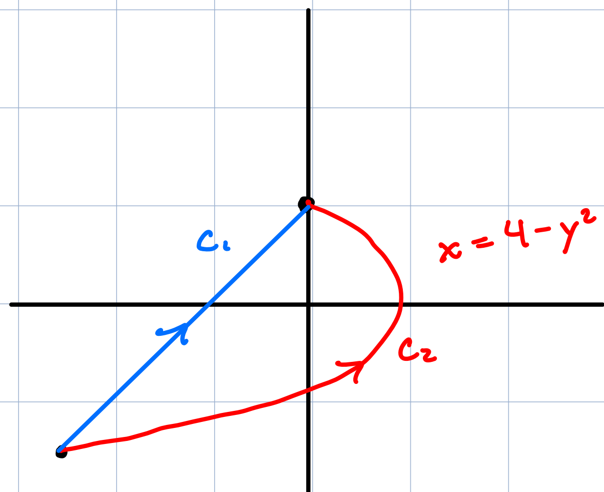
$$\int_C y^2\,dx + x\,dy = \int_C y^2\,dx + \int_C x\,dy$$

where C is either C_1 or C_2 w/

C_1 = line segment from $(-5,-3)$ to $(0,2)$

C_2 = arc $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$

Soln: ⓪ Draw



① Param curves:

$$\vec{r}_1(t) = -5(1-t)\,\vec{i} + (-3(1-t)+2t)\,\vec{j}$$
$$= (-5+5t)\,\vec{i} + (5t-3)\,\vec{j}$$

↳ $0 \le t \le 1$

$$\vec{r}_2(t) = (4-t^2)\,\vec{i} + (t)\,\vec{j}$$

↳ $-3 \le t \le 2$

② Compute:

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t-3)^2\,5\,dt + \int_0^1 (5t-5)\,5\,dt$$
$$= 5\int_0^1 25t^2 - 30t + 9 + 5t - 5\ dt$$
$$= 5\int_0^1 25t^2 - 25t + 4\ dt$$
$$= -5/6$$

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} t^2(-2t) + (4-t^2)\,dt$$
$$= \int_{-3}^{2} -2t^3 - t^2 + 4\,dt$$
$$= 40 + 5/6$$

$\Rightarrow$ line integral will generally depend on the <u>path</u> not just the end points.

Remark:

- $\int_{-C} f \, dx = - \int_C f \, dx$
- $\int_{-C} f \, dy = - \int_C f \, dy$

$\hookrightarrow$ b/c Δx_i, Δy_i will change sign when we reverse the orientation / direction of C.

Defn: Let $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field. The line integral of F along (directed) C is

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \int_a^b P(\vec{r}(t)) \cdot x'(t) + Q(\vec{r}(t)) \cdot y'(t) \quad dt$$

$$= \int_a^b F(\vec{r}(t)) \cdot r'(t) \, dt$$

$$=: \int_C F \cdot d\vec{r}$$

Example: Spse $f(x,y) = 2x^2 + 3y^2$ and C is unit circle going in counter-clockwise direction.

Compute $\int_C \nabla f \cdot d\vec{r}$.

Soln:

- $\vec{r}(t) = \cos(t)\,\vec{i} + \sin(t)\,\vec{j}$
- $\vec{r}\,'(t) = -\sin(t)\,\vec{i} + \cos(t)\,\vec{j}$
- $\nabla f = 4x\,\vec{i} + 6y\,\vec{j}$
- $\int_C \nabla f \cdot d\vec{r} = \int_0^{2\pi} (4\cos(t), 6\sin(t)) \cdot (-\sin(t), \cos(t))\, dt$

$$= \int_0^{2\pi} 2\sin(t)\cos(t)\, dt$$

$$= \sin^2(t) \Big|_0^{2\pi}$$

$$= 0$$

Defn: Given a vector field $F(x, y, z)$

$$\int_C F \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

If $F = P\vec{i} + Q\vec{j} + R\vec{k}$, then

$F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$= \big(P(r(t)) \cdot x'(t) + Q(r(t)) \cdot y'(t) + R(r(t)) \cdot z'(t)\big)\, dt$$

$$\Rightarrow \int_C F \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Example:

$$F(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{K}$$

$$\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,z \quad \text{for} \quad 0 \le t \le 1$$

$$\int_c F \cdot d\vec{r} = ?$$

Soln:

$$\int_c F \cdot d\vec{r} = \int_0^1 (t^3, t^5, t^4) \cdot (1, 2t, 3t^2)\, dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6 \quad dt$$

$$= 27/28$$

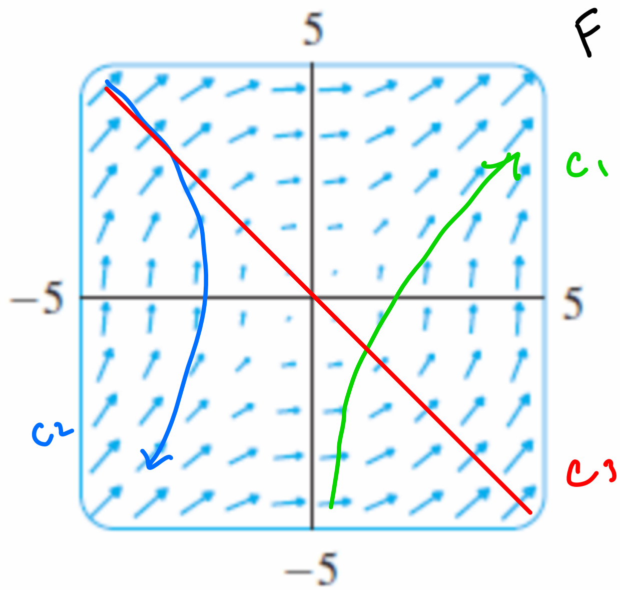
Rem:

- $\int_c F \cdot d\vec{r}$ measures how much curve "goes w/" or "goes against" the direction of F.
- $F(r(t)) \perp r'(t) \Rightarrow F(r(t)) \cdot \vec{r}'(t) = 0 \Rightarrow \int_c F \cdot d\vec{r} = 0$
- Angle between $F(r(t))$ and $\vec{r}'(t)$ is less than $90°$

  $\Rightarrow \quad F(r(t)) \cdot \vec{r}'(t) \geq 0 \Rightarrow \int_c F \cdot d\vec{r} \geq 0$
- " " " " " " greater " "

  " " $\leq 0 \Rightarrow$ " ≤ 0

Example:

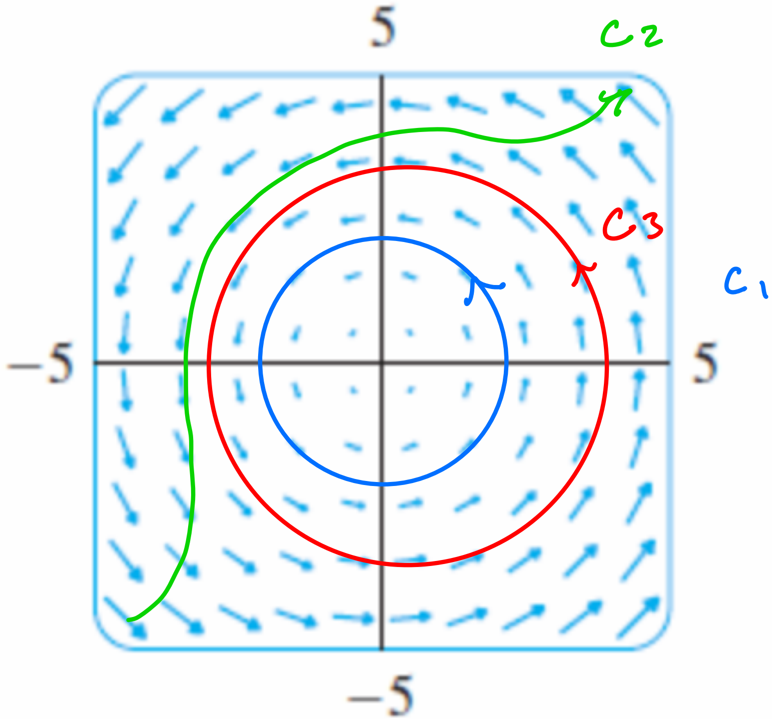


$$\int_{C_1} F \cdot d\vec{r} \geq 0$$

$$\int_{C_2} F \cdot d\vec{r} \leq 0$$

$$\int_{C_3} F \cdot d\vec{r} = 0$$

Example:

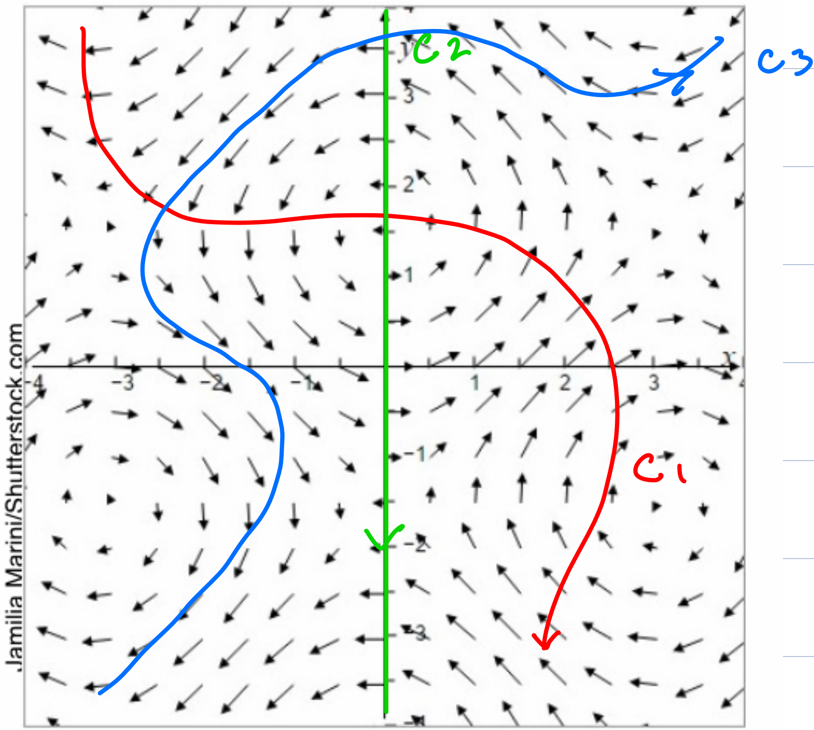


$$\int_{C_1} F \cdot d\vec{r} \leq 0$$

$$\int_{C_2} F \cdot d\vec{r} \leq 0$$

$$\int_{C_3} F \cdot d\vec{r} \geq 0$$

Example:



$$\int_{C_1} F \cdot d\vec{r} \le 0$$

$$\int_{C_2} F \cdot d\vec{r} = 0$$

$$\int_{C_3} F \cdot d\vec{r} \le 0$$

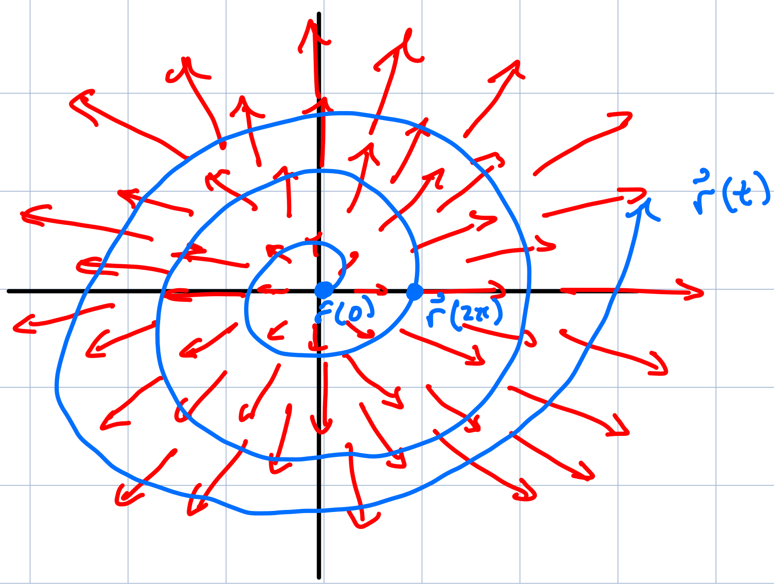
Example: $F(x,y) = x\vec{i} + y\vec{j}$.

C = spiral, $\vec{r}(t) = t\cos(t)\,\vec{i} + t\sin(t)\,\vec{j}$, $0 \le t \le 2\pi$

$\int_C F \cdot d\vec{r}$ has what sign?

Soln:

- Draw F and $\vec{r}$.



- $\vec{r}\,'(t) = \left(\cos(t) - t\sin(t)\right)\vec{i} + \left(\sin(t) + t\cos(t)\right)\vec{j}$
- $\int_0^1 \left(\cos(t), \sin(t)\right) \cdot \left(\cos(t) - t\sin(t), \sin(t) + t\cos(t)\right) dt$

$$= \int_0^{2\pi} \cos^2(t) - t\cos(t)\sin(t) + \sin^2(t) + t\sin(t)\cos(t)\ dt$$

$$= \int_0^{2\pi} 1 \ dt$$

$$= 2\pi$$