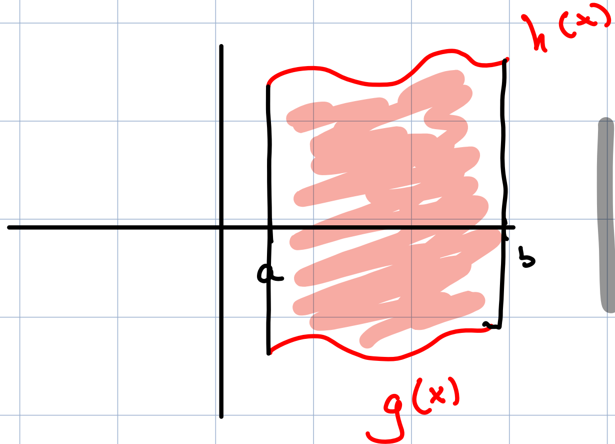
Lecture # 11

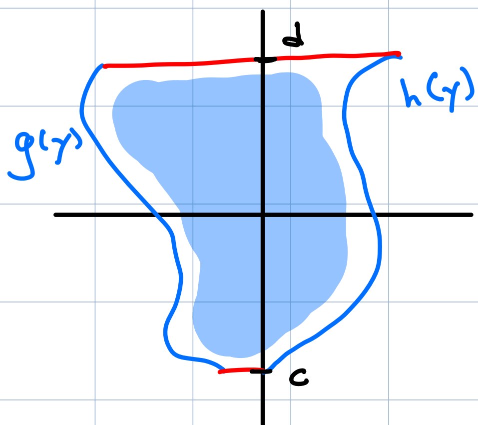
Title: Review of chapter 15 - Multivariable integration

Bounds for 2-dim'l integrals

$$\iint_R f(x,y)\,dA$$

- 

$$\Rightarrow \iint_R f\,dA = \int_a^b \int_{g(x)}^{h(x)} f \; dy\,dx$$

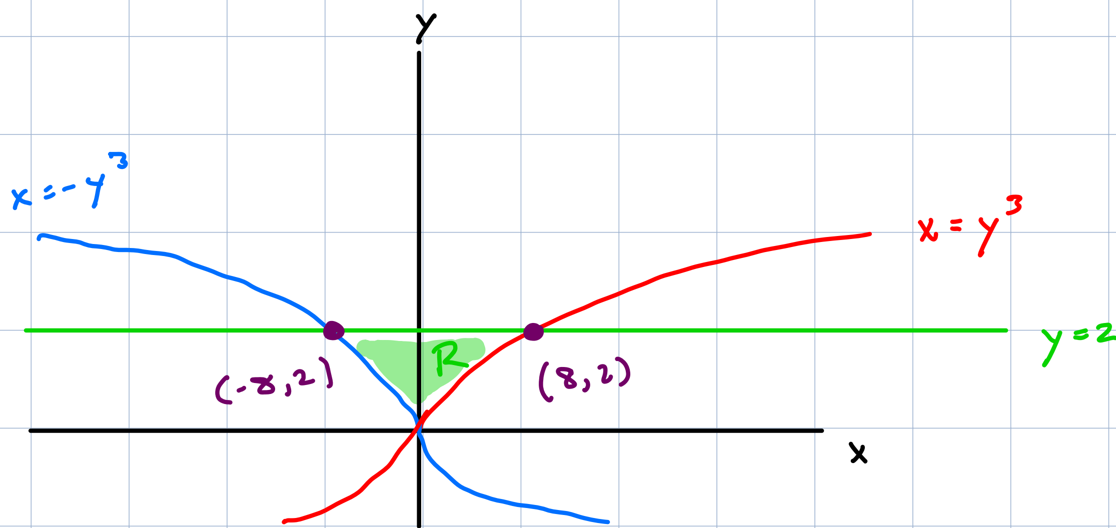
- 

$$\Rightarrow \iint_R f\,dA = \int_c^d \int_{g(y)}^{h(y)} f \; dx\,dy$$

Example: $\iint_R f\,dA$ where R is bounded by

$$x = y^3, \quad x = -y^3, \quad y = 2$$

Soln:

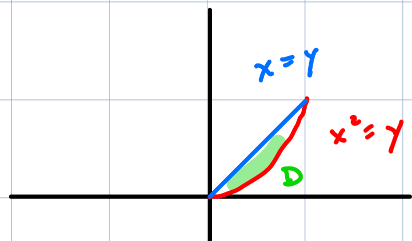


- $$\iint_R f\,dA = \int_0^2 \int_{-y^3}^{y^3} f\,dx\,dy$$
- $$\iint_R f\,dA = \int_0^8 \int_{\sqrt[3]{x}}^{2} f\,dy\,dx + \int_{-8}^{0} \int_{-\sqrt[3]{x}}^{2} f\,dy\,dx$$

Example: Compute

$$\int_0^1 \int_{x^2}^{x} \cos\left(\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right) dy\, dx$$

Soln: •



• $\int_0^1 \int_{x^2}^{x} \cos\left(\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right) dy\, dx$

$= \int_0^1 \int_y^{\sqrt{y}} \cos\left(\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right) dx\, dy$

$= \int_0^1 (\sqrt{y} - y) \cos\left(\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right) dy$

$u = \frac{2}{3}y^{3/2} - \frac{y^2}{2}$

$du = (\sqrt{y} - y)\, dy$

$= \int_0^{1/6} \cos(u)\, du$

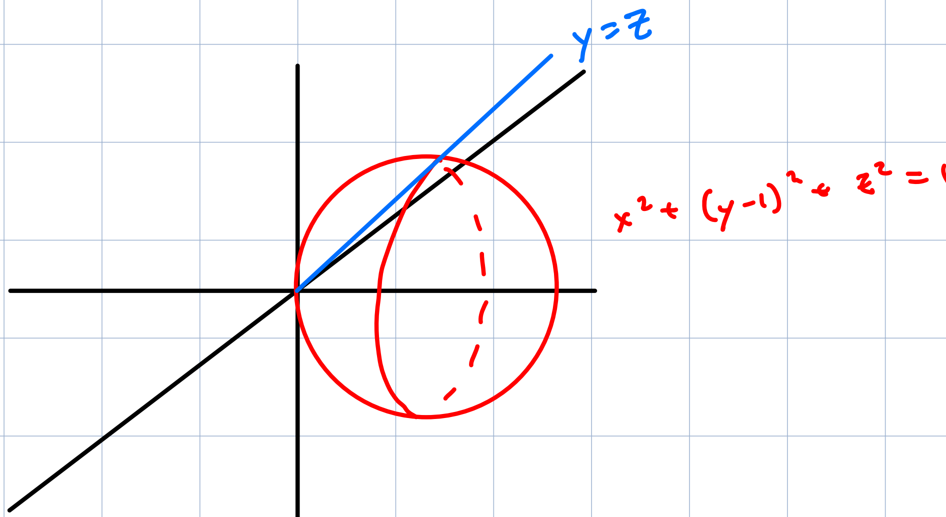
$= \sin(1/6)$

Example: Spse E is bounded by

$$x^2 + (y-1)^2 + z^2 = 1 \quad , \quad y = z$$

Compute volume using 6 different iterated integrals.

Soln: •



- lies over in xy-plane: $\qquad dz\,dx\,dy$

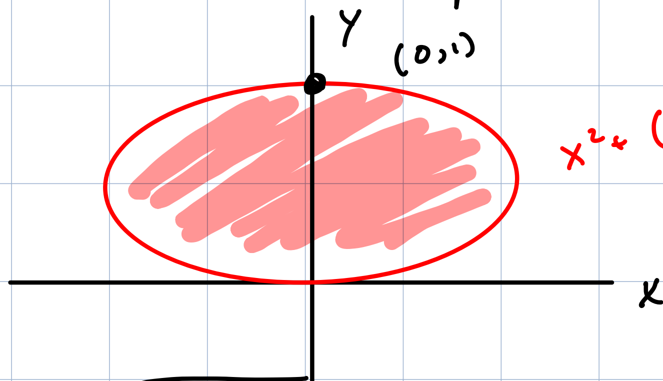
  $\hookrightarrow$ Boundary of E is intersection of

  $$x^2+(y-1)^2+z^2=1 \quad \text{and} \quad y=z$$

  Equivalently,

  $$x^2+(y-1)^2-y^2=1 \quad \text{and} \quad y=z$$

  $\Rightarrow$ lies over this ellipse

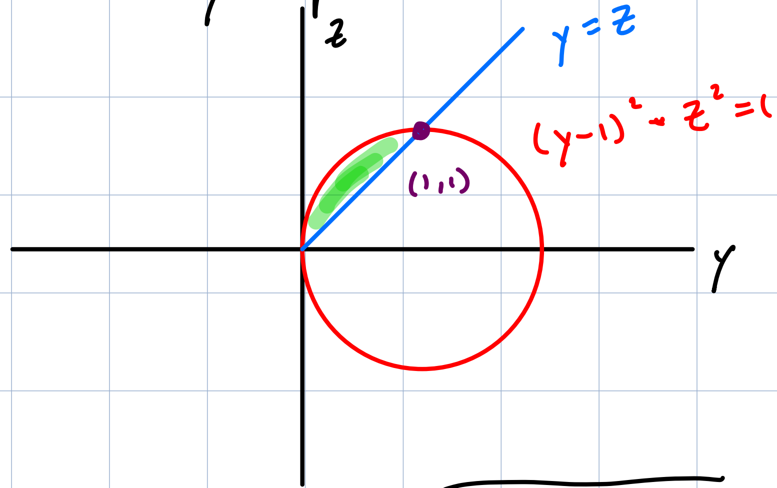
- 

  $x^2+(y-1)^2+y^2=1$

  $x=0$
  $\Rightarrow y^2-2y+1+y^2=1$
  $\Rightarrow y=0,1$

- $$\int_0^1 \int_{-\sqrt{1-y^2-(y-1)^2}}^{\sqrt{1-y^2-(y-1)^2}} \int_y^{\sqrt{1-x^2-(y-1)^2}} dz\,dx\,dy$$

- lies over in yz-plane:



- $$\int_0^1 \int_y^{\sqrt{1-(y-1)^2}} \int_{-\sqrt{1-(y-1)^2-z^2}}^{\sqrt{1-(y-1)^2-z^2}} dx\,dz\,dy$$

- lies over in xz-plane:

  $\Leftrightarrow$ Boundary of E is intersection of

  $$x^2 + (y-1)^2 + z^2 = 1 \quad \text{and} \quad y = z$$

  Equivalently,

  $$x^2 + (z-1)^2 + z^2 = 1 \quad \text{and} \quad y = z$$

  $\Rightarrow$ lies over the ellipse

- $$\int_0^1 \int_{-\sqrt{1-(z-1)^2-z^2}}^{+\sqrt{1-(z-1)^2-z^2}} \int_{1-\sqrt{1-(z-1)^2-z^2}}^{z} dy\,dx\,dz$$

Density, mass, and center of mass

Notn:
- E = region in $\mathbb{R}^3$
- $\rho(x,y,z)$ = mass per unit volume (density)

Defn:
- $\iiint_E \rho \, dV$ = mass of E wrt density ρ.
- The center of mass is the point

$$\left(\frac{\iiint_E x \cdot \rho \, dV}{\iiint_E \rho \, dV}, \frac{\iiint_E y \cdot \rho \, dV}{\iiint_E \rho \, dV}, \frac{\iiint_E z \cdot \rho \, dV}{\iiint_E \rho \, dV} \right)$$

↳ x = (x-coord of center of mass) is plane that divides mass of E into two equal pieces.

↳ Similarly for y = , z =

Cylindrical and Polar

Theorem: Spse $E = \{(x,y,z) \mid (x,y) \text{ in } D \text{ and } u_1(x,y) \leq z \leq u_2(x,y)\}$

w/ $D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta \text{ and } h_1(\theta) \leq r \leq h_2(\theta)\}$,

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} \underline{r} \cdot f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Montra: replace x w/ $r\cos(\theta)$, y w/ $r\sin(\theta)$, leave z,

dV w/ $r\,dz\,dr\,d\theta$

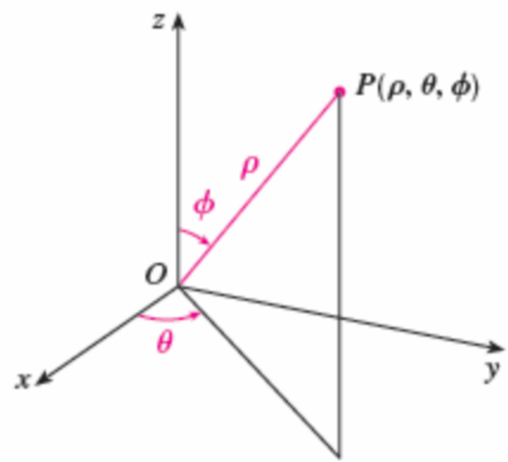
Fact: Spherical to polar

- $x = \rho \sin(\phi) \cos(\theta)$

  $y = \rho \sin(\phi) \sin(\theta)$

  $z = \rho \cos(\phi)$

  $\rho^2 = x^2 + y^2 + z^2$



Theorem: If $E = \{(\rho, \theta, \phi) \mid h_1(\theta,\phi) \leq \rho \leq h_2(\theta,\phi),\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$

$$\iiint_E f(x,y,z)\, dV$$

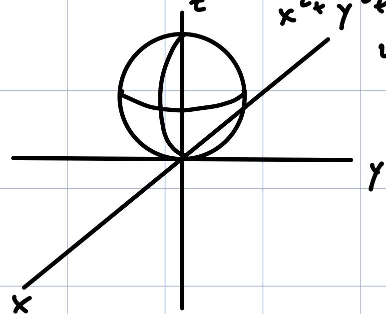
$$= \int_\alpha^\beta \int_c^d \int_{h_1(\theta,\phi)}^{h_2(\theta,\phi)} \underline{\rho^2 \cdot \sin(\phi)} f(\rho\sin(\phi)\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ d\rho\, d\phi\, d\theta$$

Example:

Question 1.3 (6 points) *Let E be the solid that is bounded by the equations $x^2+y^2+z^2=2z$, and $x=0$, and lies in the region where $x \geq 0$. Please provide an iterated triple integral expression for the quantity*

$$\iiint_E \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dV$$

either in cylindrical coordinates or in spherical coordinates. Use your iterated integral to evaluation the above integral. Provide sketches to justify your answer.

- 

  $x^2+y^2+z^2=2z \longleftrightarrow x^2+y^2+(z-1)^2=1$ w/ $x \geq 0$

- E is half of a sphere of radius 1 that is raised 1 unit in z-direction. It is divided by the yz-plane

- $x^2+y^2+z^2 \leq 2z \Rightarrow \rho \leq 2\cos(\phi)$

  $\Rightarrow \rho \leq 2\cos(\phi),\ 0 \leq \phi \leq \pi/2,\ -\pi/2 \leq \theta \leq \pi/2$

- $\iiint_E \frac{x}{(x^2+y^2+z^2)^{3/2}}\, dV$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} \int_0^{2\cos(\phi)} \frac{\rho^2 \sin(\phi)\, \rho \cos(\theta) \sin(\phi)}{\rho^3}\, d\rho\, d\phi\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} 2 \sin^2(\phi) \cos(\phi) \cos(\theta)\, d\phi\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{1} 2 u^2 \cos\theta\, du\, d\theta$$

$u = \sin\phi$

$du = \cos\phi\, d\phi$

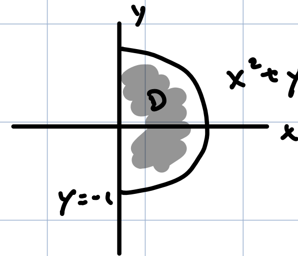
$$= \int_{-\pi/2}^{\pi/2} \frac{2}{3} \cos(\theta)\, d\theta$$

$$= \frac{4}{3}$$

- $x^2 + y^2 + (z-1)^2 \leq 1 \quad , \quad x \geq 0$

  $\Rightarrow 1 - \sqrt{1 - x^2 + y^2} \leq z \leq 1 + \sqrt{1 - x^2 - y^2}$

- E lies over D in xy-plane where



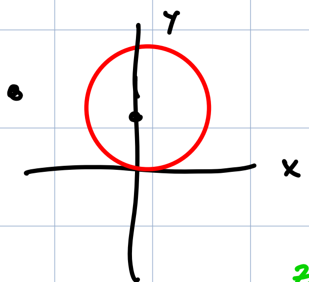
$x^2 + y^2 \leq 1 \quad , \quad x \geq 0$

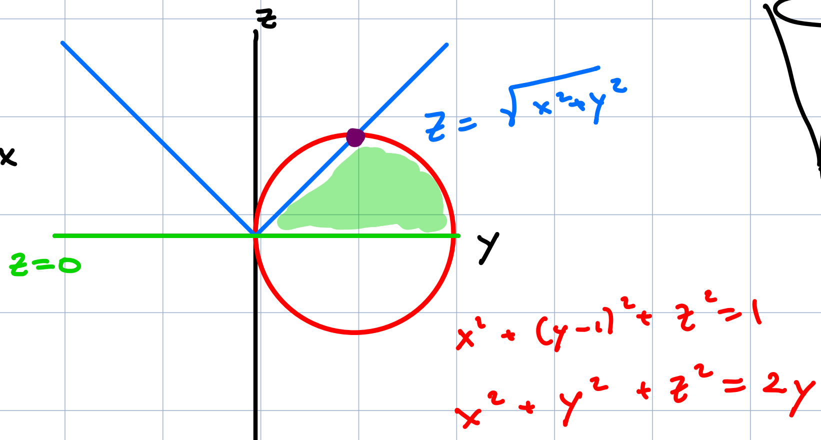
$\Rightarrow \quad 0 \leq r \leq 1 \quad , \quad -\pi/2 \leq \theta \leq \pi/2$

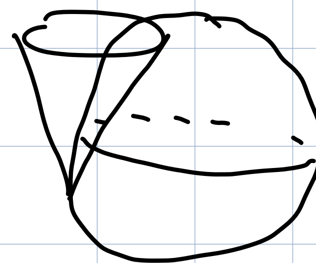
- $\displaystyle \iiint_E f \, dV = \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} r \frac{r\cos(\theta)}{(r^2+z^2)^{3/2}} \, dz \, dr \, d\theta$

Example: Set up the integral $\iiint_E zx \, dV$ in spherical coordinates where E is bounded by $x^2+(y-1)^2+z^2=1$, below $z=\sqrt{x^2+y^2}$, above $z=0$.

Soln:







- $x^2+y^2+z^2 \le 2y \Rightarrow \rho^2 \le 2\rho\sin(\phi)\sin(\theta)$

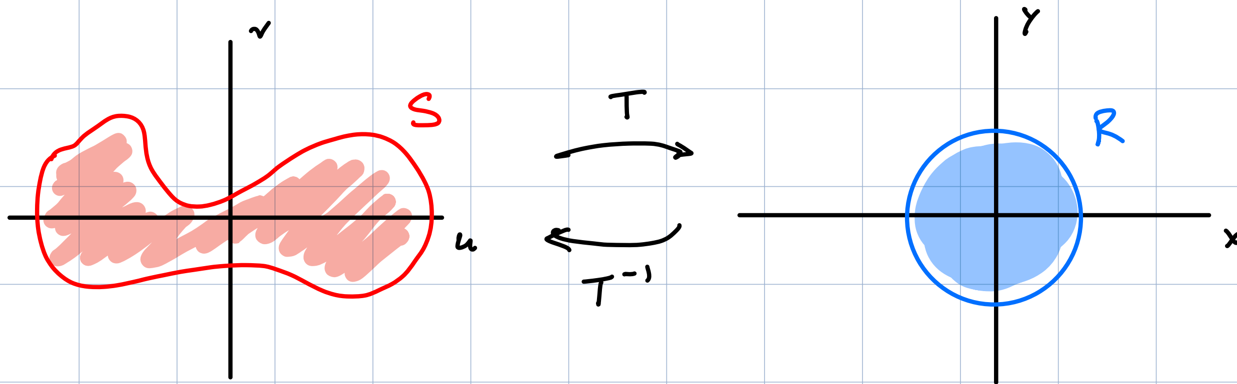
$$\Rightarrow 0 \le \rho \le 2\sin(\phi)\sin(\theta)$$

- $\pi/4 \le \phi \le \pi/2$, $0 \le \theta \le \pi$
- $\Rightarrow \iiint_E xz\,dV$

$$= \int_0^{\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\sin(\phi)\sin\theta} \rho^2 \sin\phi \cdot \rho\cos\phi\, \rho\sin\phi\cos\theta \; d\rho\, d\phi\, d\theta.$$

Intuition for Jacobians

Notn:

- $T(u,v) = \big(x(u,v), y(u,v)\big) = (x,y)$
- $T^{-1}(x,y) = \big(u(x,y), v(x,y)\big) = (u,v)$
- 

Theorem:

- $$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \cdot |J(T)|\,du\,dv$$

- $$\iint_R f(u(x,y), v(x,y))\,|J(T^{-1})|\,dA = \iint_S f(u,v)\,du\,dv$$

where

$$J(T) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

$$J(T^{-1}) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Remark: You are asked to compute an integral. What do you do?

⓪ Draw a picture

① Identify if you evaluate it w/ "older" methods (iterated, polar, cylindrical, spherical)

② If not, consider a change of variables

ⓐ Can you integrate the integrand?

↳ If not, then pick u, v to simplify your expression.

ⓑ Is your region of integration messy?

↳ If so, then pick u, v to straighten the bounds to rectangle.

(c) If both integrand and bounds are bad, then (generally) starting w/ simplifying bounds is best.

Examples:

20. $\iint_R y^2 \, dA$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$; $u = xy$, $v = xy^2$. Illustrate by using a graphing calculator or computer to draw R.

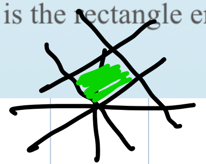
$u = xy$, $v = xy^2$ $\qquad \int_1^2 \int_1^2 \sim \text{Jac}\, dA.$

25. $\displaystyle\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, and $(0, 1)$

$u = y - x$, $v = y + x$

Example:

24. $\iint_R (x+y)\, e^{x^2-y^2}\, dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$



$1 = xy$, $x - y = 0$

$x = 4$, $x + y = 2$

Soln:

- $u = x - y$, $v = x + y$

  $\Rightarrow 0 \le u \le 2$, $0 \le v \le 3$

- $T^{-1}(x,y) = (x-y,\ x+y)$

  $$\Rightarrow J(T^{-1}) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

- $$\iint_R (x+y)\exp(x^2-y^2)\,\frac{2}{2}\,dA$$

  $2 \cdot dA = du\,dv$

  $$= \int_0^3\int_0^2 v\exp(uv)\,\frac{1}{2}\,du\,dv$$

  $$= \int_0^3 \left(\cancel{v}\exp(uv)/\cancel{v}\right)\Big|_0^2\,dv$$

  $$= \int_0^3 \exp(2v) - 1\ dv$$

  $= \text{etc.}$

$\square$