Lecture # 10

Title: Gradient vector fields + Line integrals

Section: Stewart 16.1, 16.2

Defn: A vector field on $\mathbb{R}^3$ is a fcn F that assigns to each point (x,y,z) a vector in $\mathbb{R}^3$.

$$F(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\cdot\vec{k}$$

Defn: A flowline of F is a parametric curve

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

st $F(r(t)) = \vec{r}\,'(t)$.

↪ $\vec{r}(t)$ is a path traced out by a particle being pushed by F.

Defn: The gradient vector field of $f(x,y)$ is

$$(\nabla f)(x,y) = \frac{\partial f}{\partial x}(x,y)\cdot \vec{i} + \frac{\partial f}{\partial y}(x,y)\cdot \vec{j}$$

Example:

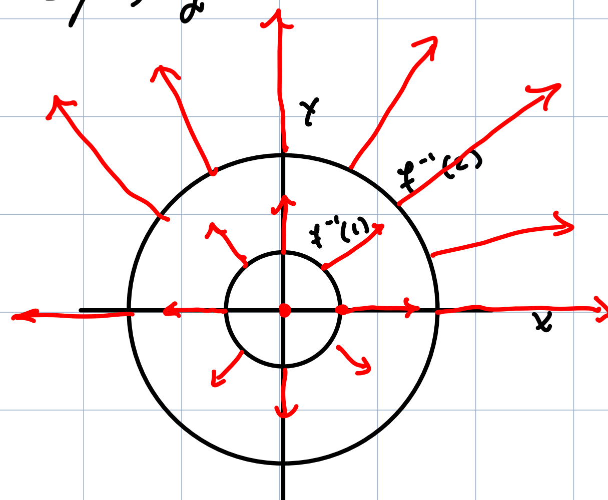
$$f(x,y) = x^2 y - y^3$$

$$(\nabla f)(x,y) = 2xy \cdot \vec{i} + (x^2 - 3y^2)\cdot \vec{j}$$

Example:

$$f(x,y) = \frac{1}{2}(x^2 + y^2)$$

$$(\nabla f)(x,y) = x\cdot \vec{i} + y \cdot \vec{j}$$



Remark: Consider a level set of f.

$$f^{-1}(c) = \{(x,y) \mid f(x,y) = c\}$$

This is (typically) some curve C in $\mathbb{R}^2$.

Fact: ∇f is always orthogonal to the level sets $f^{-1}(c)$.

Proof: Spse $f^{-1}(c)$ is parametric curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$\Rightarrow$ $r(t)$ runs along the level set of $f^{-1}(c)$

$\Rightarrow$ velocity of curve is tangent to $f^{-1}(c)$ and

$f(\vec{r}(t)) = f(x(t), y(t)) = c$ for all t

$$\Rightarrow \quad 0 = \frac{d}{dt}(c)$$

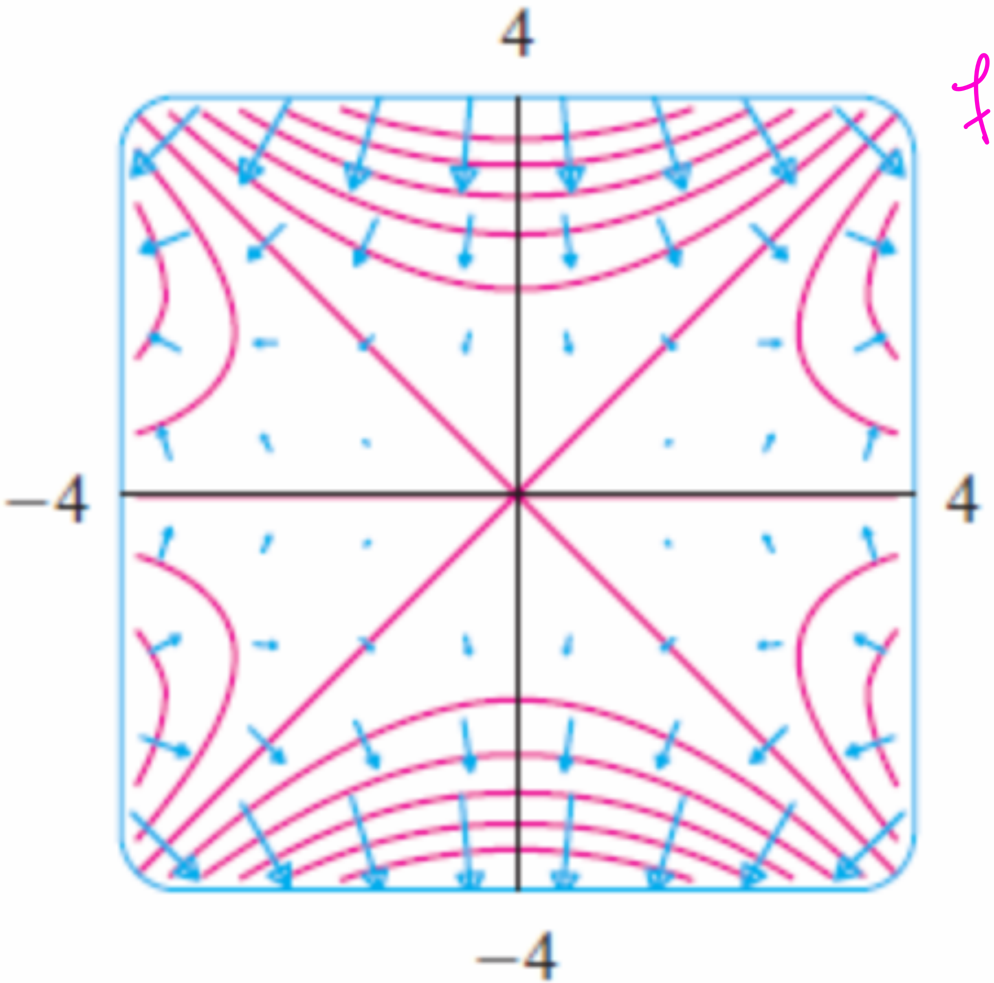
$$= \frac{d}{dt}\big(f(x(t), y(t))\big)$$

$$= f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

$$= (\nabla f)(x(t), y(t)) \bullet \vec{r}\,'(t)$$

$\Rightarrow$ $\nabla f \perp f^{-1}(c)$. $\square$

Picture :



$f^{-1}(c)$

Defn: The <u>gradient vector field</u> of $f(x,y,z)$ is

$$(\nabla f)(x,y,z) = f_x(x,y,z)\,\vec{i} + f_y(x,y,z)\,\vec{j} + f_z(x,y,z)\,\vec{k}$$

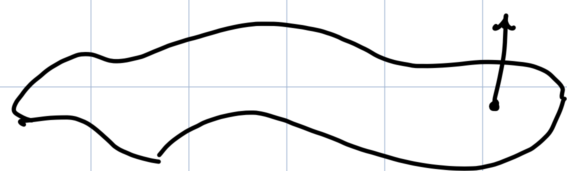
Remark: Consider a level set of f.

$$f^{-1}(c) = \{(x,y,z) \mid f(x,y,z) = c\}$$

This is (typically) some surface S in $\mathbb{R}^3$

Fact: ∇f is always orthogonal to the level sets $f^{-1}(c)$.

$\hookrightarrow$ ∇f is the normal vector to the surface



Defn: A vector F is conservative if $F = \nabla f$ for some function f. f is called a potential function for F.

Warning: Not all vector fields are conservative!

Notn: Let $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ be a parametric plane curve C w/ $a \le t \le b$.

↪ We need $\vec{r}$ to be "smooth/nice":
$\vec{r}\,'(t) \neq \vec{0}$ and $x'(t)$, $y'(t)$ are continuous

Goal: Integrate a fcn $f(x,y)$ over the curve C.

Remark: Divide $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ of width Δt_i w/ samples t_i^* in $[t_{i-1}, t_i]$.

$\vec{r}([t_{i-1}, t_i])$ divide C up into n subarcs w/ lengths Δs_i.

$f(\vec{r}(t_i^*))$

Defn: $\displaystyle\int_C f(x,y)\, ds := \lim_{n\to\infty} \sum_{i=1}^n f(x(t_i^*), y(t_i^*)) \cdot \Delta s_i$

change in arclength.

$$= \lim_{n\to\infty} \sum_{i=1}^n f(\vec{r}(t_i^*)) \cdot \left(\text{length of } \vec{r} \text{ over } [t_{i-1}, t_i] \right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^n f(\vec{r}(t_i^*)) \cdot \int_{t_{i-1}}^{t_i} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

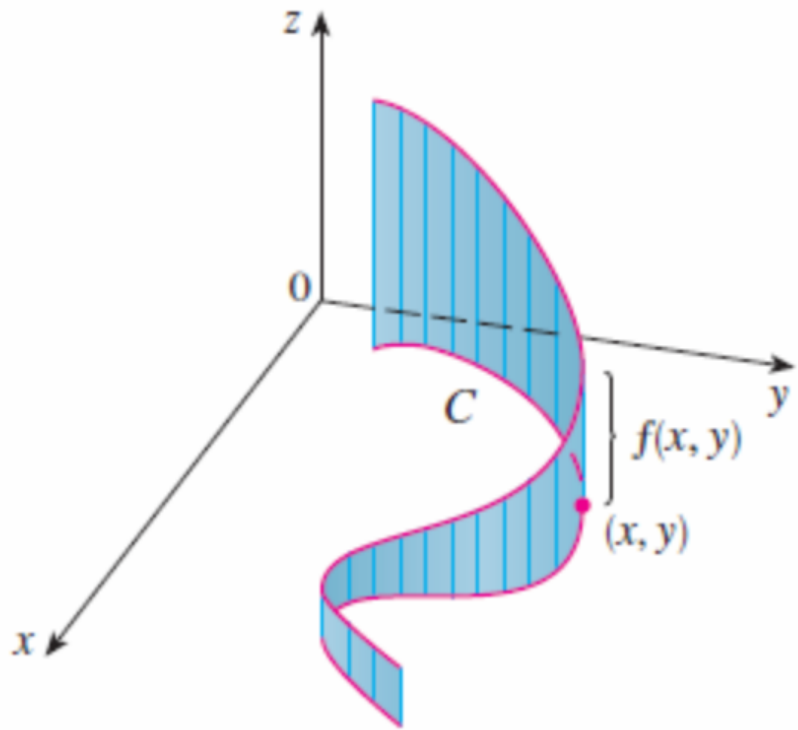
$$= \lim_{n\to\infty} \sum_{i=1}^n f(\vec{r}(t_i^*)) \cdot |r'(t_i^*)| \cdot \Delta t_i$$

$$= \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\, dt$$

Called the <u>line integral of f along C</u>

Remark:
- This does not depend on the parameterization of C as long as C is not transversed multiple times by $\vec{r}(t)$.
- This is sometype of change of coords for curves.

Remark: When $f \geq 0$, $\int_C f \, ds$ = area under "fence" whose height above (x, y) in C is $f(x, y)$.



Remark: Given $\vec{v}, \vec{w}$, the curve that starts at $\vec{v}$ and ends at $\vec{w}$ is:

$$\vec{r}(t) = t \cdot \vec{w} + (1-t) \cdot \vec{v}$$

for $0 \leq t \leq 1$.

Ex: Given $C = \{ x = y,\ 0 \le x \le 1 \}$, compute $\int_C x^2 y \, ds$

Soln:

⓪ Parameterize curve: $\vec{r}(t) = t\,\vec{i} + t\,\vec{j}$ for $0 \le t \le 1$

① Compute $\vec{r}\,'$: $\vec{i} + \vec{j}$

② Set up integral and solve:

$$\int_C x^2 y \, ds = \int_0^1 t^3 \cdot |\vec{r}\,'(t)| \, dt$$
$$= \int_0^1 t^3 \cdot \sqrt{2} \, dt$$
$$= \sqrt{2}/4$$

$\sqrt{4t^2 + 4t^2}$

$\sqrt{8t^2}$

* $\vec{r}(t) = t^2\,\vec{i} + t^2\,\vec{j}$ is another parameterization

$\vec{r}\,'(t) = 2t\,\vec{i} + 2t\,\vec{j}$

$\Rightarrow \int_C x^2 y \, ds = \int_0^1 t^6 \cdot \sqrt{8} \cdot t \, dt = \frac{\sqrt{8}}{8} = \frac{\sqrt{2}}{4}$

$\Rightarrow$ Didn't depend on parameterization.

Ex : $\int_C (2 + x^2 y)\, ds$ where C is upper half of $x^2 + y^2 = 1$

Soln :

⓪ Param. C : $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$

w/ $0 \leq t \leq \pi$

① Compute $\vec{r}'$: $\sin(t)\vec{i} - \cos(t)\vec{j}$

② Solve : $\int_C 2 + x^2 y \, ds$

$$= \int_0^{\pi} (2 + \cos^2 t \cdot \sin t) \cdot |1| \, dt$$

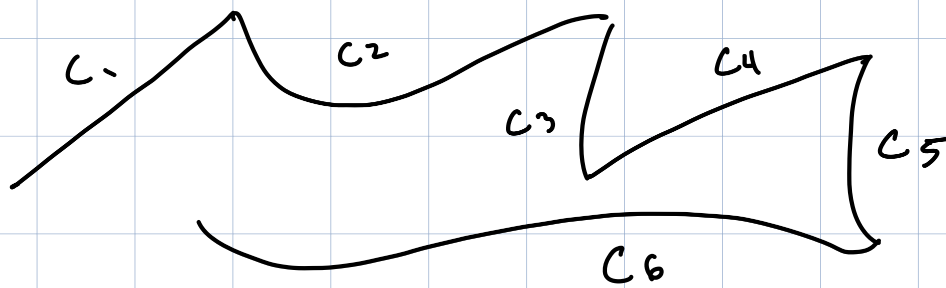
$u = \cos t$

$du = -\sin t \, dt$

$$= 2\pi - \int_1^{-1} u^2 \, du$$

$$= 2\pi + \frac{2}{3}$$

Defn: A piecewise smooth curve, C, is a union of a finite # of smooth curves $C_1, \dots, C_n$ w/



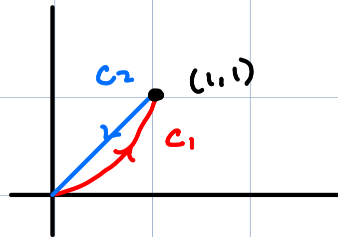
Defn: C = piecewise smooth curve:

$$\int_C f\, ds = \sum_{i=1}^{n} \int_{C_i} f\, ds$$

Ex: $\int_C 2x \, ds$ where C_1 is $y = x^2$ from $(0,0)$ to $(1,1)$

and C_2 is line segment from $(1,1)$ to $(0,0)$

Soln: ⓪ Draw pic:



① Param curves: $\vec{r}_1(t) = t\,\vec{i} + t^2\,\vec{j}$

$\vec{r}_2(t) = (1-t)\,\vec{i} + (1-t)\,\vec{j}$

② Compute $|\vec{r}\,'|$: $\vec{r}_1\,' = \vec{i} + 2t\,\vec{j}$

$\vec{r}_2\,' = -\vec{i} - \vec{j}$

③ Set up integrals and solve

$u = 1 + 4t^2$
$du = 8t\,dt$

$$\int_c 2x\,ds = \int_0^1 2t \cdot \sqrt{1+4t^2}\,dt + 2\int_0^1 \sqrt{2}\cdot(1-t)\,dt$$

$$= \int_1^5 \frac{1}{4}\sqrt{u}\,dt + 2\int_0^1 \sqrt{2}\,(1-t)\,dt$$

$$= \frac{2}{3}\cdot\frac{1}{4}\,u^{3/2}\Big|_1^5 + 2\left(\sqrt{2}\,t - \frac{\sqrt{2}}{2}t^2\right)\Big|_0^1$$

$$= \frac{1}{6}\sqrt{125} - \frac{1}{6} + 2\sqrt{2} - \sqrt{2}$$

Fact: If $-C$ = curve C but w/ opposite direction, then

$$\int_c f\,ds = \int_{-c} f\,ds$$

"Proof": $\int_c f\,ds$ = "signed" area under C raised up by f

= " " " $-C$ " " " f

$$= \int_{-c} f\,ds$$

$\square$

Notn:

- Let $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$ be a parametric curve in $\mathbb{R}^3$ for some curve C and $a \le t \le b$.
- $\vec{r}\,'(t) = x'\,\vec{i} + y'\,\vec{j} + z'\,\vec{k}$ is the velocity of r.

Defn:

- $\int_C f(x,y,z)\,ds$

$$= \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

- Length of $C = \int_C 1\,ds$
- $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

$$= \int_a^b P(r(t))\,x'(t) + Q(r(t))\,y'(t) + R(\vec{r}(t))\,z'(t)\,dt$$

Ex: Spse $\vec{r}(t) = (\cos(t), \sin(t), t)$ = helix for $0 \le t \le 2\pi$.

Evaluate $\int_C y \sin(z)\, ds$

Soln: $$\int_C y \sin(z)\, ds = \int_0^{2\pi} \sin^2(t) \cdot \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\; dt$$

$$= \int_0^{2\pi} \sqrt{2} \sin^2(t)\, dt$$

$$= \frac{\sqrt{2}}{2} \int_0^{2\pi} 1 - \cos(2t)\; dt$$

$$= \sqrt{2} \cdot \pi$$