Lecture # 1

Title: Double Integrals over Rectangles

Section: Stewart 15.1

Review of the definite integral

Notation:

- $f : [a, b] \longrightarrow \mathbb{R}$
- Divide $[a, b]$ into subintervals $[x_{i-1}, x_i]$

  w/ $x_i - x_{i-1} = (b-a)/n = \Delta x$
- Sample/pick x_i^* in $[x_{i-1}, x_i]$
- Reimann sum is $\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$

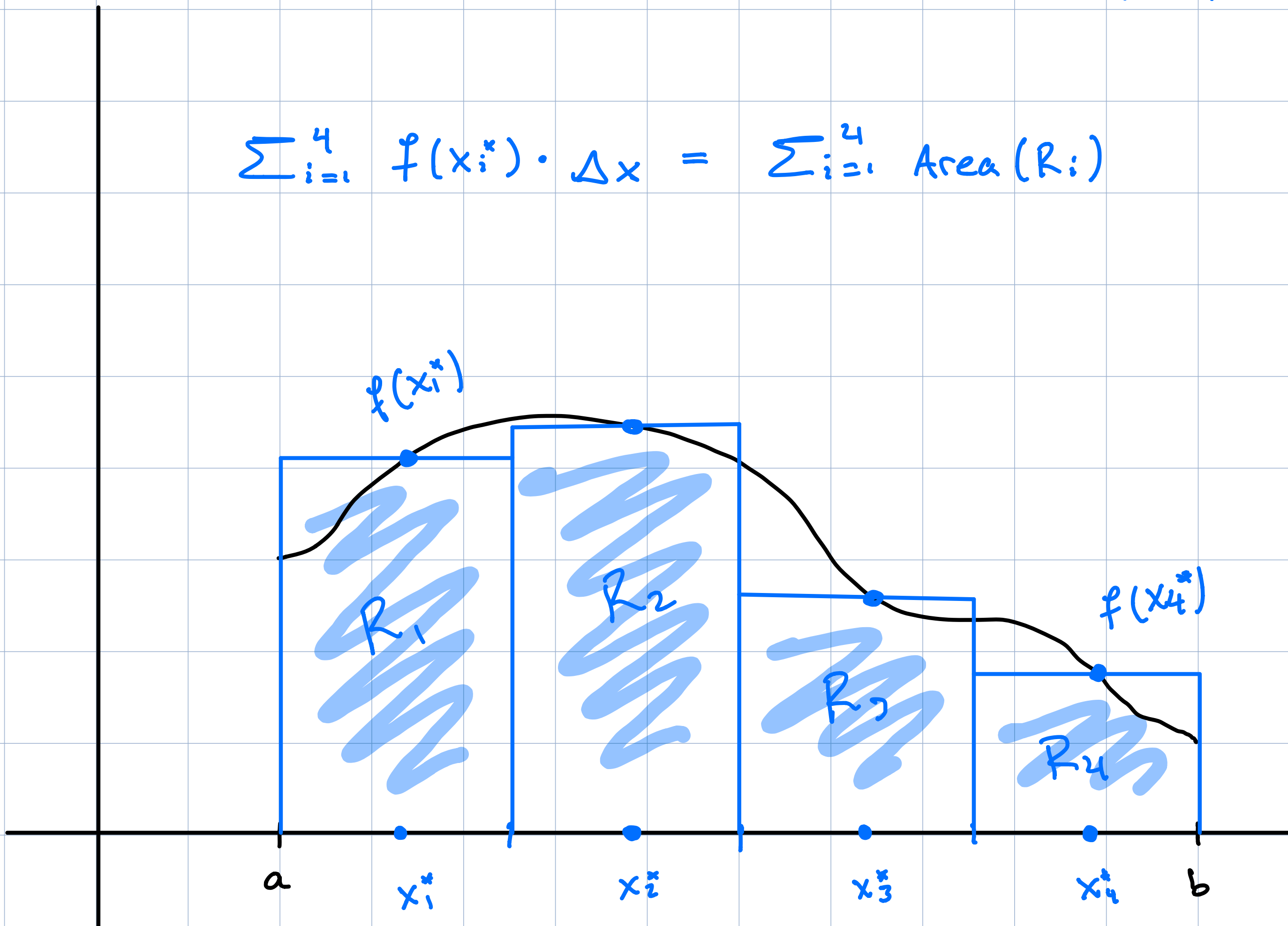
Definition: $\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$

Remark: $\int_a^b f(x)\, dx$ when $f > 0$ is the <u>area</u> under the graph of f.

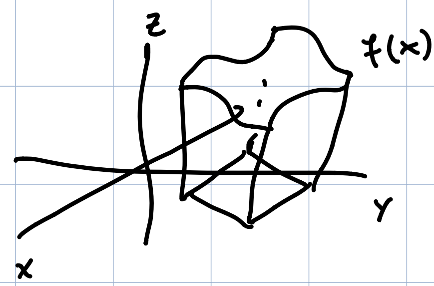
Picture:

$n = 4$

$$\sum_{i=1}^{4} f(x_i^*) \cdot \Delta x = \sum_{i=1}^{4} \text{Area}(R_i)$$



Volume and the double integral



Notation: • $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$

Question: What is the volume under the graph of f?

Notation: • Divide $[a,b]$ into subintervals $[x_{i-1}, x_i]$

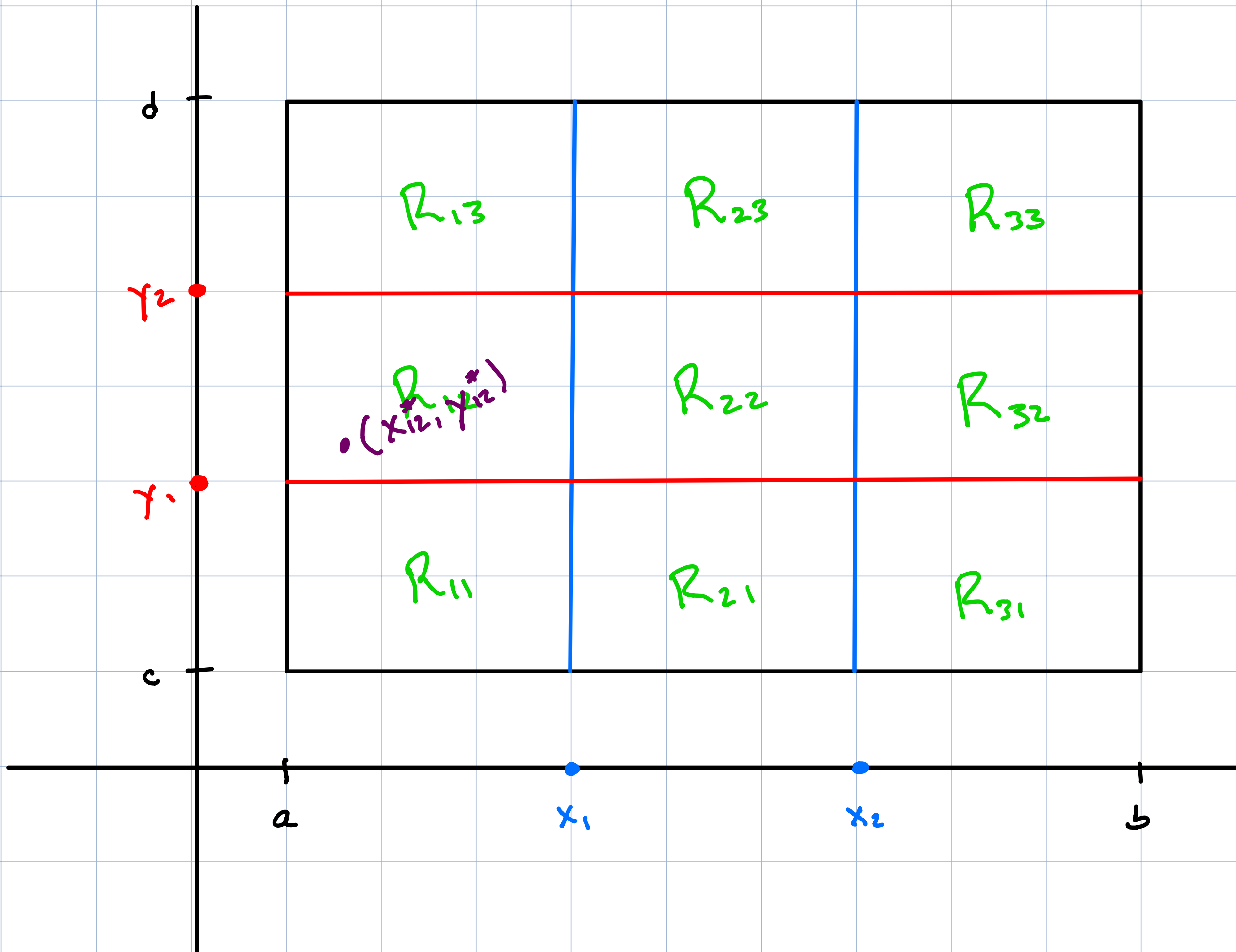
w/ $x_i - x_{i-1} = \Delta x = (b-a)/n$

• " $[c,d]$ " " $[y_{i-1}, y_i]$

" $y_i - y_{i-1} = \Delta y = (d-c)/n$

• $R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$

Picture:



Notation:

- $\text{Area}(R_{ij}) = \Delta x \cdot \Delta y = \Delta A$
- Sample/pick (x_{ij}^*, y_{ij}^*) in R_{ij}
- Reimann sum is $\sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$

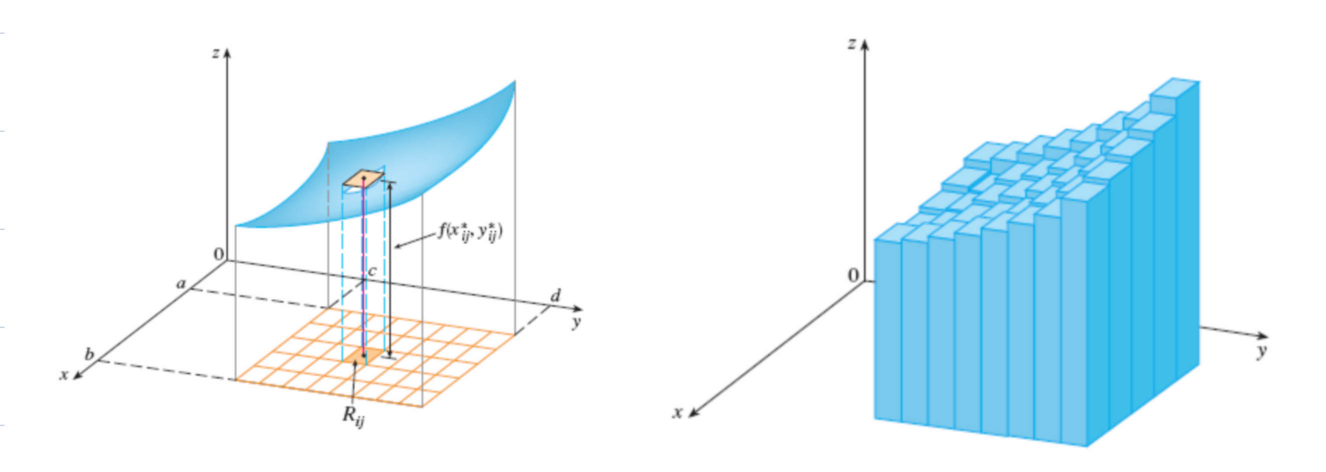
Definition:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$$

Notation:

$$\iint_R f(x,y)\,dA \coloneqq \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Picture:



$f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$ = Area of column over R_{ij}.

Remark: As $n \to \infty$, $\sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$ approximates volume under the graph of f when $f \geq 0$

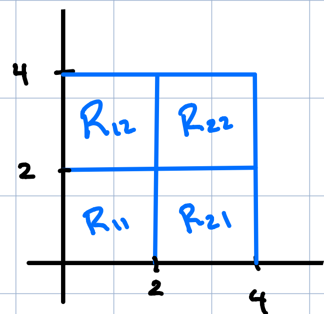
$\Rightarrow \int_c^d \int_a^b f(x,y)\,dx\,dy$ = Volume under the graph of f that lies above R

Example: Estimate the volume under the graph of

$$f(x,y) = 100 - 2\cdot x^3 + 4y^2$$

over $R = [0,4] \times [0,4]$

Solution: ① Divide R



$\rightsquigarrow$ $\text{Area}(R_{ij}) = 4$

② Sample: $(1,1)$ in R_{11} $\rightsquigarrow$ $f(1,1) = 102$

$(1,3)$ " R_{12} $\rightsquigarrow$ $f(1,3) = 134$

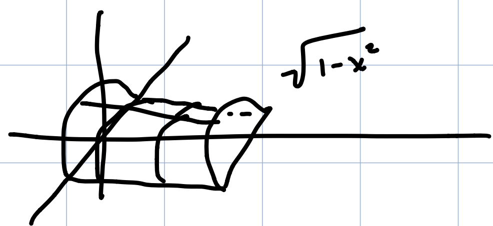
$(3,1)$ " R_{21} $\rightsquigarrow$ $f(3,1) = 50$

$(3,3)$ " R_{22} $\rightsquigarrow$ $f(3,3) = 118$

③ Sum: $4(102 + 134 + 50 + 118) = 1616$ (?)

Example: Compute $\int_0^2 \int_{-1}^1 \sqrt{1-x^2}\, dx\, dy$.

Solution: ① Realize: Integral as volume under $\sqrt{1-x^2}$



$\rightsquigarrow$ volume of half of a cylinder of

$\rightsquigarrow$ volume of half of cylinder of height 2 and radius 1 $= \pi(1)^2 \cdot 2/2 = \pi$

② Use volume formula: $\int_0^2 \int_0^2 \sqrt{1-x^2}\, dx\, dy = \pi$

Iterated Integrals

Question: How do we compute double integrals

↪ **Answer:** Reduce to (iterated) single integrals.

Notation:

$$A(x) = \int_c^d f(x,y)\,dy$$

↪ we integrate f where we view x as a constant

Example:

$$A(x) = \int_0^2 (x^2 + xy^3)\,dy =$$

$$= x^2 y \Big|_{0=y}^{2=y} + xy^4/4 \Big|_{y=0}^{2}$$

$$= 2x^2 + 4x$$

Remark:
- Now integrate A: $\int_a^b A(x)\,dx$
- We have iterated our integration

$$\underbrace{\int_a^b \left[\underbrace{\int_c^d f(x,y)\,dy}_{\text{integrate over } y} \right] dx}_{\text{integrate over } x}$$

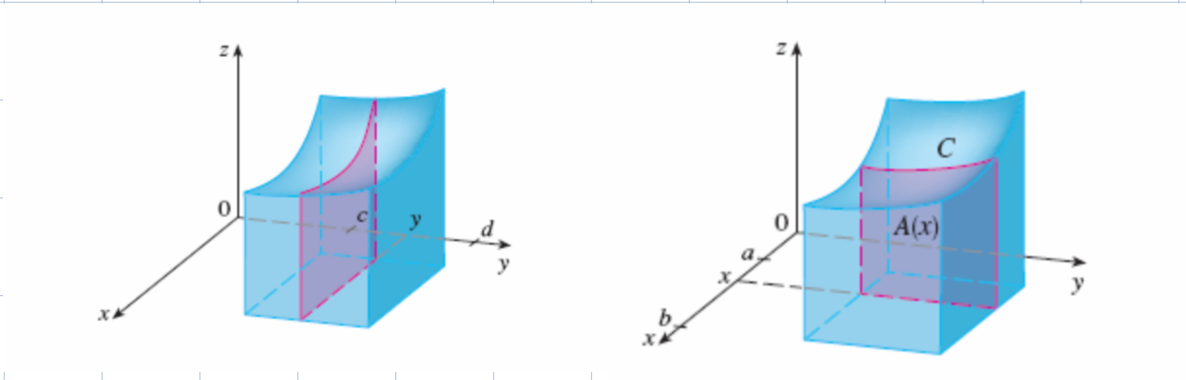
Theorem:

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Idea:
$$\iint f \, dy \, dx = \int (\text{Area of } y\text{-slices}) \, dx$$
$$= \text{"volume under } f\text{"}$$
$$= \int (\text{Area of } x\text{-slices}) \, dy$$
$$= \iint f \, dx \, dy$$

Picture:



Example: $\int_0^2 \int_0^{\pi} x^2y + \cos(x)\, dx\, dy$

① Integrate $\int_0^{\pi} x^2y + \cos(x)\, dx$ w/ "y" = "constant"

$$\int_0^{\pi} x^2y + \cos(x)\, dx = \left(x^3y/3 + \sin(x)\right)\Big|_{x=0}^{x=\pi}$$

$$= \pi^3 y/3$$

② Integrate $\int_0^2 \pi^3 y/3\, dy$ normally

$$\int_0^2 \pi^3 y/3\, dy = \left(\pi^3 y^2/6\right)\Big|_{y=0}^{2}$$

$$= \pi^3 \cdot 4/6$$

Remark: We can ✓ other direction

$$\int_0^{\pi} \left(\int_0^2 x^2 y + \cos(x) \; dy \right) dx$$

$$= \int_0^{\pi} \left(\frac{x^2 y^2}{2} + y\cos(x) \right) \Big|_{y=0}^{2} dx$$

$$= \int_0^{\pi} \left(2x^2 + 2\cos(x) \right) dx$$

$$= \left(\frac{2x^3}{3} + 2\sin(x) \right) \Big|_0^{\pi}$$

$$= \frac{2}{3} \pi^3$$

Exercise:

Evaluate $\int_0^{\pi}\int_0^{3} y\sin(yx)\,dy\,dx$

$$\int_0^{3}\int_0^{\pi} y\sin(yx)\,dx\,dy$$

$$= \int_0 \left(-\cancel{y}\cos(yx)/\cancel{y}\right)\Big|_{x=0}^{\pi}\,dy$$

$$= \int_0^{3} 1 - \cos(\pi y)\,dy$$

$$= \left(y - \sin(\pi y)/\pi\right)\Big|_{y=0}^{3}$$

$$= 3$$

Remark: Sometimes one order is <u>easier</u>!

Exercise: Volume of solid contained by $3x^2 + 4y^2 + z = 64$, $x = 1$ plane, $y = -2$ plane, and the three coordinate planes?

① Phrase as integral problem.

$3x^2 + 4y^2 + z = 64$ = graph of $f(x,y) = 64 - 3x^2 - 4y^2$

$\Rightarrow$ Volume $= \displaystyle\int_{-2}^{0}\int_{0}^{1} 64 - 3x^2 - 4y^2 \, dx\, dy$

② Compute $= \displaystyle\int_{-2}^{0} \left(64x - x^3 - 4y^2x\right)\Big|_0^1 \, dy$

$= \displaystyle\int_{-2}^{0} 64 - 1 - 4y^2 \, dy$

$= \left(63y - \frac{4}{3}y^3\right)\Big|_{y=-2}^{0}$

$=$ etc.

Average Value

Definition: $f : [a,b] \longrightarrow \mathbb{R}$, the average value of f is

$$\text{Average} = \frac{1}{(b-a)} \cdot \int_a^b f(x)\,dy$$

Definition: $f : \overset{R}{\overset{\shortparallel}{[a,b] \times [c,d]}} \longrightarrow \mathbb{R}$, the average value of f

$$\text{Ave.} = \frac{1}{\text{Area}(R)} \cdot \iint_R f(x,y)\,dA$$

Example: What is the average value of ye^{-yx} over $[0,1] \times [-1,1]$?

Solution: ① Compute area of region

↳ Area $= 2$

② Compute integral over region then divide by area

$$\int_{-1}^{1} \int_{0}^{1} y e^{-xy} \, dx \, dy$$

$$= \int_{-1}^{1} \left(-y e^{-xy}/y \right) \Big|_0^1 \, dy$$

$$= \int_{-1}^{1} \left(-e^{-x} + 1 \right) dy$$

$$= \left(e^{-x} + x \right) \Big|_{-1}^{1}$$

$$= e^{-1} + 1 - e^{1} + 1$$

$$\Rightarrow \text{Aver.} = \left(e^{-1} - e^{1} + 2 \right) / 2$$