18.901 - Homework 9

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1 (5 points). *Which of the following subsets define subgroups. Justify your answers.*

(i) $GL_n(\mathbb{R}) \subseteq GL_n(\mathbb{C})$.

(ii) $(\{\pm 1\}, \times) \subseteq (\mathbb{R} \setminus 0, \times)$.

(iii) $\mathbb{Z}_{\geq 0} \subseteq (\mathbb{Z}, +)$.

(iv) $\mathbb{R}_{>0} \subseteq (\mathbb{R} \setminus 0, \times)$.

(v) $\left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \neq 0 \right\} \subseteq GL_2(\mathbb{R})$.

Question 2 (5 points). *Let G be a group and let $g, h \in G$. Show that the orders of gh and hg agree.*

Question 3 (5 points). *Consider*

$$G = \left\{ A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{R}) \right\}.$$

Let $\varphi\colon G \to (\mathbb{R} \setminus 0, \times)$ be given by $\varphi(A) = a^2$.

(i) *Show that φ is a homomoprhism.*

(ii) *Compute the kernel of φ.*

(iii) *Compute the image of φ.*

Question 4 (5 points). *Let $\varphi\colon G \to G'$ be a surjective homomorphism.*

(i) *Show that if G is cyclic, then G' is cyclic.*

(ii) *Show that if G is abelian, then G' is abelian.*

Question 5 (5 points). *Let $H \subseteq G$ be a subgroup of a finite order group G. Show that $|H|$ divides $|G|$.*

Question 6 (5 points). *Let $\varphi\colon G \to G'$ be a group homomorphism such that G has finite order. Derive a relationship between the orders of G, $\ker(\varphi)$ and $\mathrm{Im}(\varphi)$.*

Question 7 (5 points). *Let $\varphi\colon G \to G'$ be a non-trivial group homomorphism. Suppose that $|G| = 18$ and $|G'| = 15$. What are the possible orders of the kernel of φ?*

Question 8 (5 points). *Prove that every group of order 21 contains an element of order 3.*

Question 9 (5 points). *Consider the subsets*

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{R}) \right\}$$

and

$$K = \left\{ \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{R}) \right\}.$$

(i) *Show that H is a subgroup of $GL_3(\mathbb{R})$.*

(ii) *Show that K is a normal subgroup of H.*

(iii) *Identify the quotient group H/K.*

Question 10 (5 points). *Let $G \subseteq GL_n(\mathbb{R})$ be a subgroup.*

(i) *Show that if* $A, B, C, D \in G$ *and if there exist paths in* G *from* A *to* B *and from* C *to* D*, then there is a path in* G *from* AC *to* BD*.*

(ii) *Show that the set of matrices in* G *that can be joined to the identity matrix form a normal subgroup of* G*.*

Question 11 (5 points)**.** *Given two spaces* X *and* Y*, show that*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, x_0),$$

where the right-hand-side is the product group (see the notes for a definition).

Question 12 (5 points)**.** *Show that for a space* X*, the following three conditions are equivalent:*

(i) *Every map* $S^1 \to X$ *is homotopic to a constant map.*

(ii) *Every map* $\alpha\colon S^1 \to X$ *extends to a map* $A\colon D^2 \to X$ *such that* $\alpha = A|_{\partial D^2}$*.*

(iii) $\pi_1(X, x_0)$ *is the trivial group with one element for all* $x_0 \in X$*.*