18.901 - Homework 8

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1 (5 points). *Let X and Y be homeomorphic spaces. Show that X has covering dimension n if and only if Y has covering dimension n.*

Question 2 (5 points). *Let X be a space with covering dimension less than or equal to n. Let $A \subseteq X$ be a closed subspace. Show that A has covering dimension less than or equal to n.*

Question 3 (5 points). *Let X be a manifold. Show that there exists a continuous function $f\colon X \to \mathbb{R}$ such that for every compact subset $K \subseteq \mathbb{R}$, $f^{-1}(K) \subseteq X$ is compact.*

Question 4 (5 points). *Show that $[0,1]$ can not be written as a countable union of closed subsets each having empty interior.*

Question 5 (10 points). *Let $M_{m,n}(\mathbb{R})$ denote the space of m-by-n matrices with real entries and let $\det\colon M_{n,n}(\mathbb{R}) \to \mathbb{R}$ denote the determinant function.*

(i) Show that $\det^{-1}(\mathbb{R} \smallsetminus 0)$ is open.

(ii) Show that matrix multiplication $f\colon M_{m,n}(\mathbb{R}) \to M_{n,k}(\mathbb{R})$ given by $f(X) = XA$ for $A \in M_{n,k}(\mathbb{R})$ is continuous.

(iii) Show that $\det^{-1}(\mathbb{R} \smallsetminus 0)$ is dense in $M_{n,n}(\mathbb{R})$. (Hint: Use item (2) above and a singular value decomposition.)

(iv) Consider the map $\mathcal{D}\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$ given by

$$\mathcal{D}\left(\begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}\right) = \prod_{i_1 < \cdots < i_m} \det\begin{pmatrix} x_{i_1} & \cdots & x_{i_m} \end{pmatrix},$$

where $x_i \in \mathbb{R}^m$. Show that $\mathcal{D}^{-1}(\mathbb{R} \smallsetminus 0)$ is dense and open.

Question 6 (5 points). *(i) Show that if $f\colon X \to Y$ is a homotopy equivalence and $g\colon Y \to Z$ is a homotopy equivalence, then $g \circ f$ is a homotopy equivalence.*

(ii) Show that if $f\colon X \to Y$ is homotopic to $g\colon X \to Y$ rel $A \subseteq X$ and g is homotopic to $h\colon X \to Y$ rel $A \subseteq X$, then f is homotopic to h rel $A \subseteq X$.

Question 7 (5 points). *Construct an explicit deformation retraction of $\mathbb{R}^n \smallsetminus \{0\}$ onto S^{n-1}.*

Question 8 (5 points). *Give an example of a space X and a subspace $A \subseteq X$ such that X retracts onto A but X does not deformation retract onto A.*

Question 9 (5 points). *Suppose that $r\colon X \to X$ is a retraction onto a subspace $A \subseteq X$. Show that if X is contractible, then A is contractible.*

Question 10 (5 points). *Show that X is contractible if and only if every map $f\colon X \to Y$ (for arbitrary Y) is null-homotopic if and only if every map $g\colon Y \to X$ (for arbitrary Y) is null-homotopic.*

Question 11 (5 points). *Suppose that $f\colon X \to Y$ is a homotopy equivalence. Show that f gives a bijection between the path-components of X and the path-components of Y.*