18.901 - Homework 7

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1 (5 points)**.** *Let M be a n-manifold and let $f\colon M \to \mathbb{R}^n$ be a continuous map. Show that*

$$\Gamma(f) = \{(x,y) \in M \times \mathbb{R}^n \mid f(x) = y\}$$

is a n-manifold.

Question 2 (5 points)**.** *Give an open cover of S^n by two open subsets that are both homeomorphic to $\mathbb{R}^n$.*

Question 3 (5 points)**.** *Consider $\mathbb{C}^n$ and notice that $\mathbb{C}^n = \mathbb{R}^{2n}$ as a set. Endow $\mathbb{C}^n$ with the standard topology on $\mathbb{R}^{2n}$. Define an equivalence relation on $\mathbb{C}^n \setminus \{0\}$ by $z \sim w$ if and only if $z = \lambda \cdot w$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Show that $\mathbb{CP}^n := (\mathbb{C}^n \setminus \{0\})/\sim$ is a $(2n-2)$-manifold.*

Question 4 (5 points)**.** *Let $M_{n,n}(\mathbb{R})$ denote the set of n-by-n matrices with real entries. Notice that $M_{n,n}(\mathbb{R}) = \mathbb{R}^{n^2}$ and equip it with the standard topology. Let $GL(n) \subset M_{n,n}(\mathbb{R})$ denote the subset of invertible matrices. Is $GL(n)$ a manifold? Either prove it or disprove it.*

Question 5 (5 points)**.** *Suppose that X is* locally n-Euclidean, *that is, for each $x \in X$ there is an open neighborhood $x \in U$ such that U is homeomorphic to $\mathbb{R}^n$.*

(i) Show that if X is compact and Hausdorff, then X is a manifold.

(ii) Show that the converse statement does not hold.

Question 6 (5 points)**.** *Let X be a paracompact space and let Y be a compact Hausdorff space. Show that $X \times Y$ is paracompact.*

Question 7 (5 points)**.** *Let X be a paracompact space. Let $A \subseteq X$ be a closed subspace. Show that A is paracompact.*

Question 8 (5 points)**.** *Let M be a m-manifold. Let $N \subseteq M$ be a subset such that for every $x \in N$ there exists an open subset $U \subseteq M$ and a homeomorphism $\psi\colon \mathbb{R}^m \to U$ such that $x \in U$ and $\psi|_{\mathbb{R}^n \times \{0\}^{m-n}}$ defines a homeomorphism onto $N \cap U$.*

(i) Show that N is a n-manifold.

(ii) Show that there exists a continuous function $f\colon M \to \mathbb{R}$ such that $f|_N \equiv 0$ and $f|_{M \setminus N} > 0$.

Question 9 (5 points)**.** *Let X be a Hausdorff space. Suppose that there exist compact subsets $K_n \subset X$ such that $K_n \subset \operatorname{int}(K_{n+1})$ and $X = \cup_n K_n$. Show that X is normal.*