18.901 - Homework 6

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1 (5 points)**.** *Show that if Y is a compact space, then the projection*

$$pr_X \colon X \times Y \to X, \quad pr_X(x, y) = x$$

is closed*, that is, $pr_X(C)$ is closed for all closed subsets $C \subseteq X \times Y$. (Hint: Show that $X \smallsetminus pr_X(C)$ is open for all $C \subset X \times Y$ closed.)*

Question 2 (5 points)**.** *Let $f \colon X \to Y$ be a continuous map, where Y is a compact Hausdorff space. Show that f is continuous if and only if the graph of f,*

$$\Gamma(f) \coloneqq \{(x, f(x)) \in X \times Y\},$$

is closed in $X \times Y$.

Question 3 (5 points)**.** *Let $f \colon X \to Y$ be a closed continuous surjective map such that $f^{-1}(y)$ is compact for all $y \in Y$.*

(i) Show that if X is Hausdorff, then Y is Hausdorff.
(ii) Does the above conclusion still hold if f is no longer surjective?
(iii) Does the above conclusion still hold if $f^{-1}(y)$ is not compact for some $y \in Y$.

Question 4 (5 points)**.** *Let $f \colon X \to Y$ be a closed continuous surjective map such that $f^{-1}(y)$ is compact for all $y \in Y$.*

(i) Show that if X is second countable, then Y is second countable.
(ii) Does the above conclusion still hold if f is no longer closed?

Question 5 (5 points)**.** *Let $f \colon X \to Y$ be a closed continuous surjective map.*

(i) Show that if X is normal, then Y is normal.
(ii) Does the above conclusion still hold if f is no longer closed?

Question 6 (5 points)**.** *Let $f \colon X \to Y$ be a closed continuous surjective map such that $f^{-1}(y)$ is compact for all $y \in Y$.*

(i) Show that if Y is compact, then X is compact.
(ii) Does the above conclusion still hold if f is no longer surjective?

Question 7 (5 points)**.** *Give a direct proof of Urysohn's lemma for metric spaces.*

Question 8 (5 points)**.** *(i) Show that every compact metric space is second countable.*
(ii) Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

Question 9 (5 points)**.** *If X is a Hausdorff space with a countable basis, does it necessarily follow that X is metrizable?*