18.901 - Homework 5

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1. *Let $X = \mathbb{R}$ with the topology in which open subsets are the complements of finitely many points (and the empty set). Is X compact? Either prove or disprove it.*

Question 2. *Let $M_{n,n}(\mathbb{R})$ denote the set of n-by-n matrices with real entries. Notice that $M_{n,n}(\mathbb{R}) = \mathbb{R}^{n^2}$ and equip it with the standard topology. Let $O(n) \subseteq M_{n,n}(\mathbb{R})$ denote the subset of orthogonal matrices. Let $GL(n) \subseteq M_{n,n}(\mathbb{R})$ denote the subset of invertible matrices.*

(i) Is $O(n)$ connected? Either prove it or disprove it.
(ii) Is $O(n)$ compact? Either prove it or disprove it.
(iii) Is $GL(n)$ compact? Either prove it or disprove it.

Question 3. *Let $f\colon X \to Y$ be a continuous injective map. Show that if Y is Hausdorff, then X is Hausdorff.*

Question 4. *Let A and B be two disjoint compact subsets of a Hausdorff space X. Show that there exist open subsets U and V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$.*

Question 5. *Let $f\colon X \to Y$ be a continuous map. Suppose that X is compact and Y is Hausdorff. Show that f is* closed*, that is, $f(C)$ is closed for every closed subset $C \subseteq X$.*

Question 6. *Let $X = \mathbb{R} \times \{0, 1\}$ with the equivalence relation $\sim$ given by $(x, i) \sim (x, i)$ for all (x, i) and $(x, 0) \sim (x, 1)$ if $x \neq 0$. Let $Y = X/\sim$ denote the quotient.*

(i) Show that Y is T_1.
(ii) Show that Y is not Hausdorff.

Question 7. *Show that the Tietz extension theorem implies Urysohn's lemma.*

Question 8. *Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous functions. Show that if Y is Hausdorff, then*

$$\{x \in X \mid f(x) = g(x)\}$$

is closed in X.

Question 9. *Let X and Y be T_1 spaces. Show that if the product $X \times Y$ is normal, then X and Y are normal.*

Question 10. *Let X be a T_1 space. Suppose that A and B are closed normal subspaces of X. Does it follow that $A \cup B$ is normal? Either prove it or disprove it.*