18.901 - Homework 4

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1 (5 points)**.** *Let A be a connected subspace of X.*

(i) Does it necessarily follow that $\operatorname{int}(A)$ is connected?

(ii) Does it necessarily follow that $\partial(A)$ is connected?

In each case, provide either a proof or a counter-example.

Question 2 (5 points)**.** *Let $A \subseteq X$ be a subset of a topological space. Show that if C is a connected subspace of X such that $C \cap A \neq \varnothing$ and $C \cap (X \setminus A) \neq \varnothing$, then $C \cap \partial A \neq \varnothing$.*

Question 3 (10 points)**.** *Let X and Y be topological spaces. Let A be a proper subset of X and let B be a proper subset of Y.*

(i) Does it necessarily follow that X and Y are connected if and only if $(X \times Y) \setminus (A \times B)$ is connected.

(ii) Does it necessarily follow that X and Y are path-connected if and only if $(X \times Y) \setminus (A \times B)$ is path-connected.

In each case, provide either a proof or a counter-example.

Question 4 (5 points)**.** *Let $f\colon X \to Y$ be a continuous map. Let $C(x)$ and $P(x)$ denote the connected component and path-component respectively that each contain x.*

(i) Show that $f(C(x)) \subset C(f(x))$.

(ii) Show that $f(P(x)) \subset P(f(x))$.

Question 5 (5 points)**.** *Let X_1 and X_2 be connected spaces and fix points $x_1 \in X_1$ and $x_2 \in X_2$. Set $A = \{x_1, x_2\}$. Show that the quotient $(X_1 \sqcup X_2)/A$ is connected.*

Question 6 (5 points)**.** *Let $q\colon X \to Y$ be a quotient map. Show that if $q^{-1}(y)$ is connected for all $y \in Y$ and Y is connected, then X is connected.*

Question 7 (5 points)**.** *Let X be a space. Show that each connected component of X is closed.*

Question 8 (5 points)**.** *Let X be space that admits a basis $\mathcal{B}$ such that each $B \in \mathcal{B}$ is connected.*

(i) Show that each connected component of X is open.

(ii) Show that X is (not just as a set!) given as the topological disjoint union of its connected components.

Question 9 (5 points)**.** *Let X be a space that admits a basis $\mathcal{B}$ such that each $B \in \mathcal{B}$ is path-connected. Show that if X is connected, then X is path-connected. (Hint: Show that for each $x \in X$ that $P(x)$ is open.)*