18.901 - Homework 3

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Definition. *A set map $f\colon X \to Y$ of spaces is* open *(resp.* closed*) if for each open (resp. closed) subset $U \subseteq X$, $f(U)$ is open (resp. closed).*

Question 1. *Let $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ be homeomorphisms. Show that*

$$f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2, \quad (f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$$

is a homeomorphism.

Question 2. *Let X and Y be spaces and let $f\colon X \to Y$ be a map. Define*

$$f(X) = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}.$$

Consider $f(X) \subseteq Y$ with the subspace topology. Show that f is continuous if and only if $f\colon X \to f(X)$ is continuous.

Question 3. *Let X be a topological space. Show that if $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous functions, then $f - g\colon X \to \mathbb{R}$, $f \cdot g\colon X \to \mathbb{R}$, and $f + g\colon X \to \mathbb{R}$ are all continuous functions. Moreover, if $g \neq 0$, show that $f/g\colon X \to \mathbb{R}$ is a continuous function. (Hint: Can you reduce to the case of $X = \mathbb{R}$ with the standard topology and results from class? Question 2 may also be helpful.)*

Question 4. *Let $f\colon X \to Y$ be a homeomorphism. Let $A \subseteq X$ be a subset.*

(i) Describe the closure, interior, and boundary of $f(A)$.

(ii) Endow A with the subspace topology. Show that the restriction of f to A defines a homeomorphism from A to $f(A)$, where $f(A) \subseteq Y$ is endowed with the subspace topology. (Hint: Some question above is helpful.)

Question 5. *Let*

$$\mathbb{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}.$$

and

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

(i) Show that $\partial \mathbb{D}^n = S^{n-1}$.

(ii) Show that $\operatorname{int}(\mathbb{D}^n)$ is homeomorphic to $\mathbb{R}^n$.

Question 6. *Define*

$$C^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in [-1, 1]\}.$$

(i) Show that C^n is homeomorphic to $\mathbb{D}^n$.

(ii) Show that $\operatorname{int}(C^n)$ is homeomorphic to $\operatorname{int}(\mathbb{D}^n)$.

(iii) Show that ∂C^n is homeomorphic to S^{n-1}.

Question 7. *Endow $\mathbb{R}^{n+1} \setminus (0, \ldots, 0) \subseteq \mathbb{R}^{n+1}$ with the subspace topology, where $\mathbb{R}^{n+1}$ is equip with the standard topology. Equip $S^n \times (0, +\infty)$ with the product topology. Show that $\mathbb{R}^{n+1} \setminus (0, \ldots, 0) \cong S^n \times (0, +\infty)$. (Hint: Polar coordinates?)*

Question 8. *Let $f\colon X \to Y$ be an open surjective continuous map.*

(i) Show that if X is first countable, then $f(X)$ is first countable.

(ii) Conclude that if X is homeomorphic to Y, then X if first countable if and only if Y is first countable.

Question 9. *Define*

$$\mathcal{X} = \{\mathbb{Z}_{\geq 0} \setminus \{x_1, \ldots, x_n\} \mid x_i \in \mathbb{Z}_{\geq 0} \text{ and } n \in \mathbb{Z}_{\geq 0}\}.$$

Let $Y = \mathbb{Z}^2_{\geq 0}$ and define a topology on Y as follows: $U \subseteq Y$ is open if and only if either

(i) $(0,0) \notin U$ or

(ii) $(0,0) \in U$ and U contain $\cup_{s \in X} (\{s\} \times X_s)$ for some $X \in \mathcal{X}$ and $X_s \in \mathcal{X}$.

Let $A = \mathbb{Z}^2_{\geq 1} \subseteq \mathbb{Z}^2_{\geq 0} = Y$.

(i) Show that $(0,0) \in \overline{A}$.

(ii) Show that there does not exist a sequence of points $y_n \in A$ such that $y_n \to (0,0)$.

(iii) Conclude that Y is not first countable.