18.901 - Homework 2

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1. *Let $(X, \mathcal{O})$ be a topological space. Suppose that $\mathcal{B}$ is a basis that generates $\mathcal{O}$. Show that $U \in \mathcal{O}$ if and only if U is a union of elements of $\mathcal{B}$.*

Question 2. *Consider*

$$\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}.$$

(i) *Show that $\mathcal{B}$ defines a basis for $\mathbb{R}$.*
(ii) *Show that the topology generated by $\mathcal{B}$ agrees with the standard topology on $\mathbb{R}$.*

Question 3. *Let $\mathcal{O}$ be the topology on $\mathbb{R}$ given by the subsets $\varnothing$,$\mathbb{R}$, and $(a, +\infty)$ for all $a \in \mathbb{R}$.*
(i) *Show that $\mathcal{O}$ defines a topology on $\mathbb{R}$.*
(ii) *Given $A \subseteq X = (\mathbb{R}, \mathcal{O})$, describe $\overline{A}$.*

Question 4. *Suppose that A is a subset of a topological space X. Show $X = \operatorname{int}(A) \sqcup \partial A \sqcup \operatorname{int}(X \smallsetminus A)$.*

Question 5. *Suppose that A is a subset of a topological space X.*
(i) *Show that if $\partial A = \varnothing$, then A is open and closed.*
(ii) *Show that A is open if and only if $\partial A = \overline{A} \smallsetminus A$.*
(iii) *If A is open, does it necessarily follow that $A = \operatorname{int}(\overline{A})$?*

Question 6. *Let A_α for $\alpha \in \mathcal{A}$ be a (possibly uncountable) collection of subsets of X.*
(i) *Show that $\overline{\cup_\alpha A_\alpha} \supseteq \cup_\alpha \overline{A_\alpha}$.*
(ii) *Show that $\overline{\cap_\alpha A_\alpha} \subseteq \cap_\alpha \overline{A_\alpha}$.*
(iii) *Show that the opposite inclusions in (i) and (ii) do not hold in general.*

Question 7. *Let A_α for $\alpha \in \mathcal{A}$ be a (possibly uncountable) collection of subsets of X.*
(i) *Show that $\operatorname{int}(\cap_\alpha A_\alpha) \subseteq \cap_\alpha \operatorname{int}(A_\alpha)$.*
(ii) *Show that $\operatorname{int}(\cup_\alpha A_\alpha) \supseteq \cup_\alpha \operatorname{int}(A_\alpha)$.*
(iii) *Show that the opposite inclusions in (i) and (ii) do not hold in general.*

Question 8. *Let X be a topological space and let $A \subseteq X$ be a subspace.*
(i) *Show that if $B \subseteq A$ is open in the subspace topology of A and $A \subseteq X$ is open in X, then B is open in X.*
(ii) *Show that if $B \subseteq A$ is closed in the subspace topology of A and $A \subseteq X$ is closed in X, then B is closed in X.*

Question 9. *Let X be a topological space and let $Y \subseteq X$ be a subspace. Let $A \subseteq Y \subseteq X$ be a subset. Show that the closure of A in Y with respect to the subspace topology of Y is equal to the intersection of Y with the closure of A in X.*

Question 10. *Give an example of a set X and two different metrics $d\colon X \times X \to \mathbb{R}$ and $d'\colon X \times X \to \mathbb{R}$ such that the metric topologies determined by d and d' do not agree. (Hint: The discrete topology.)*

Questions 11, 12, and 13 are suggested, but not required to be turned in. They will not be graded.

Question 11. *Suppose that $U \subseteq X$ is open and $A \subseteq X$ is closed.*

(i) *Show that $U \smallsetminus A$ is open.*

(ii) *Show that $A \smallsetminus U$ is closed.*

Question 12. *Let A and B be subsets of a topological space X.*

(i) *Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

(ii) *Show that $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.*

Question 13. *Suppose that A is a subset of a topological space X.*

(i) *Show that $\overline{X \smallsetminus A} = X \smallsetminus \operatorname{int}(A)$.*

(ii) *Show that $\operatorname{int}(X \smallsetminus A) = X \smallsetminus \overline{A}$.*