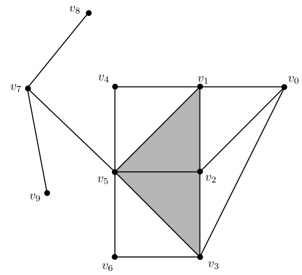
18.901 - Homework 14

Spring 2025

Please read and follow the instructions below.

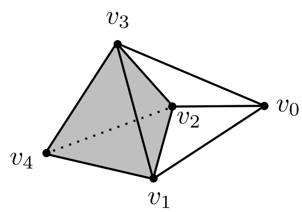
- We have listed the problems according to their relevance for the final exam.

Question 1. *Consider the following simplicial complex:*



(i) Compute the Betti numbers of the simplicial complex.
(ii) Compute the homology groups of the simplicial complex, providing explicit bases.

Question 2. *Consider the following simplicial complex:*



The 3-simplex depicted is not "filled-in", that is, the simplicial complex is composed of a copy of $\partial\Delta^3$ and three additional edges.
(i) Compute the Betti numbers of the simplicial complex.
(ii) Compute the homology groups of the simplicial complex, providing explicit bases.

Question 3. *Let K_n denote the connected sum of n-copies of $\mathbb{RP}^2$. Compute $H_\bullet(K_n)$.*

Question 4. *Show that every continuous map $f\colon S^n \to S^m$ is null-homotopic rel a basepoint whenever $n < m$.*

Question 5. *Suppose that $p\colon \Sigma_h \to \Sigma_g$ is a fibre bundle with discrete fibre F. What is the relationship between h and g?*

Question 6. *Consider the Klein bottle K viewed as a quotient of the square where the horizontal sides are identified and the vertical sides are "oppositely" identified. Let $\alpha\colon S^1 \to K$ denote the loop that corresponds to the image of the horizontal sides in the quotient and let $\beta\colon S^1 \to K$ denote the loop that corresponds to the image of the vertical sides in the quotient.*
(i) Let $X_\alpha = T^2 \cup_{\{0\}\times S^1=\alpha} K$. Compute $H_\bullet(X_\alpha)$.
(ii) Let $X_\beta = T^2 \cup_{\{0\}\times S^1=\beta} K$. Compute $H_\bullet(X_\beta)$.

Question 7. *Let X be the space that is obtained from identifying the north and south poles of S^2 to a single point.*
(i) Show that X is homotopy equivalent to a wedge sum of some familiar spaces.
(ii) Compute $H_\bullet(X)$.

Question 8. *Consider the space X obtained from S^2 by attaching n copies of $\mathbb{D}^2$ along n disjoint circles in S^2.*
(i) Show that $X \simeq \bigvee_{i=1}^{n+1} S^2$.
(ii) Compute $H_\bullet(X)$.

Question 9. *Let $f_\bullet \colon C_\bullet \to D_\bullet$, $g_\bullet \colon C_\bullet \to D_\bullet$, and $h_\bullet \colon C_\bullet \to D_\bullet$ denote chain maps.*

(i) Show that f is chain homotopic to itself.

(ii) Show that if f is chain homotopic to g, then g is chain homotopic to f.

(iii) Show that if f is chain homotopic to g and g is chain homotopic to h, then f is chain homotopic to h.