18.901 - Homework 13

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1. *Consider an abstract simplicial complex K. Show that $|C(K)|$ is homeomorphic to $(|K| \times I)/(|K| \times \{0\})$.*

Question 2. *Consider an abstract simplicial complex K. Show that $|S(K)|$ is homeomorphic to $(|K| \times I)/\sim$, where $\sim$ is the equivalence relation generated by the relations $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in |K|$.*

Question 3. *Consider abstract simplicial complexes K and L. Show that $|K \vee L|$ is homeomorphic to $|K| \vee |L|$.*

Question 4. *Consider the simplicial complex Δ^n that is the geometric realization of the abstract simplicial complex given by $\mathcal{P}(\{0, \ldots, n\})$.*

(i) Show that the subspace $\partial\Delta^n \subseteq \Delta^n$ corresponds to the geometric realization of a subcomplex of $\mathcal{P}(\{0, \ldots, n\})$.
(ii) Show that Δ^n is homeomorphic to $\mathbb{D}^n$.
(iii) Show that $\partial\Delta^n$ is homeomorphic to S^{n-1}.

Question 5. *Consider the chain complex*

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} -1 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}} \mathbb{R}^5 \xrightarrow{\begin{pmatrix} 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 \end{pmatrix}} \mathbb{R}^2,$$

where the final term $\mathbb{R}^2$ is C_0.

(i) Show that the above sequence of maps is, in fact, a chain complex.
(ii) Compute the Betti numbers of the chain complex.
(iii) Compute the homology groups of the chain complex, providing a basis for each homology group.

Question 6. *Let $(C'_\bullet, \partial'_\bullet)$ and $(C''_\bullet, \partial''_\bullet)$ be two chain complexes. Define*

$$C_\bullet = C'_\bullet \oplus C''_\bullet \qquad \partial_\bullet = \begin{pmatrix} \partial'_\bullet & 0 \\ 0 & \partial''_\bullet \end{pmatrix}.$$

(i) Show that $(C_\bullet, \partial_\bullet)$ is a chain complex.
(ii) Show that the projection $pr' \colon C_\bullet \to C'_\bullet$ is a chain map.
(iii) Show that $(pr')_* \times (pr'')_* \colon H_n((C_\bullet, \partial_\bullet)) \to H_n((C'_\bullet, \partial'_\bullet)) \oplus H_n((C''_\bullet, \partial''_\bullet))$ is an isomorphism.

Question 7. *Consider a bounded chain complex $(C_\bullet, \partial_\bullet)$. Define a new chain complex by $D_\bullet = C_{-\bullet}$ with boundary map given by $\partial_\bullet^T$.*

(i) Show that $(D_\bullet, \partial_\bullet^T)$ defines a chain complex.
(ii) Establish a relationship between the Betti numbers of $(D_\bullet, \partial_\bullet^T)$ and $(C_\bullet, \partial_\bullet)$.

Question 8. *Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a chain map of chain complexes. For the below items, either provide a proof of the statement or provide a counter-example.*

(i) If f_i is injective for all i, then the induced map $(f_i)_* \colon H_i(C) \to H_i(D)$ is injective.
(ii) If f_i is surjective for all i, then the induced map $(f_i)_* \colon H_i(C) \to H_i(D)$ is surjective.
(iii) If f_i is an isomorphism for all i, then the induced map $(f_i)_* \colon H_i(C) \to H_i(D)$ is an isomorphism.

Question 9. *Let* $(C_\bullet, \partial_\bullet)$ *be a chain complex. For the below items, either provide a proof of the statement or provide a counter-example.*

(i) If the Euler characteristic of $(C_\bullet, \partial_\bullet)$ *is zero, then* $H_i(C) = 0$ *for all* i*.*

(ii) If the sequence associated to the chain complex is a long exact sequence, then $H_i(C) = 0$ *for all* i*.*

(iii) If there exists a (self-) chain map $f_\bullet \colon C_\bullet \to C_\bullet$ *such that* $\operatorname{tr}(f_i) \neq 0$ *for all* i*, then there exists some* j *such that* $H_j(C) \neq 0$*.*

(iv) If $f_\bullet \colon C_\bullet \to C_\bullet$ *and* $g_\bullet \colon C_\bullet \to C_\bullet$ *are two chain maps such that* $(f_i)_* = (g_i)_*$ *for all* i*, then* $f_i = g_i$ *for all* i*.*

Question 10. *Consider the sequence*

$$0 \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0.$$

(i) Show that if the sequence is exact at A*, then* φ *is injective.*

(ii) Show that if the sequence is exact at B*, then* φ *is surjective.*

Question 11. *Consider the following commutative diagram of vectors spaces with exact rows:*

$$\begin{tikzcd}
V_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & V_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"'] & V_3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"'] & V_4 \arrow[d, "\alpha_4"'] \\
W_1 \arrow[r, "g_1"] & W_2 \arrow[r, "g_2"] & W_3 \arrow[r, "g_3"] & W_4.
\end{tikzcd}$$

(i) Show that if α_1 *is surjective and* α_2 *and* α_4 *are injective, then* α_3 *is injective.*

(ii) Show that if α_4 *is injective and* α_1 *and* α_3 *are surjective, then* α_2 *is surjective.*

(iii) Using the above results, prove the 5-lemma.