18.901 - Homework 12

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

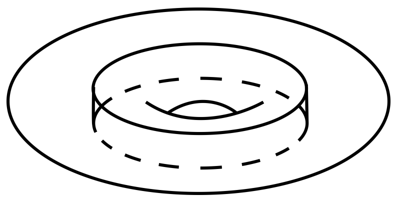
Question 1. *Let X be a finite set of points in $\mathbb{R}^n$ for $n \geq 2$. Compute the fundamental group of the complement, $\pi_1(\mathbb{R}^n \smallsetminus X)$.*

Question 2. *Let X be n straight lines through the origin in $\mathbb{R}^m$ for $m \geq 3$. Compute the fundamental group of the complement, $\pi_1(\mathbb{R}^m \smallsetminus X)$.*

Question 3. *Let X be the space that is obtained from identifying the north and south poles of S^2 to a single point. Compute $\pi_1(X)$.*

Question 4. *Let X be the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus. Compute $\pi_1(X)$.*

Question 5. *Consider the space X depicted below:*



In words, X is obtained from gluing a cylinder inside a torus in such a manner that the gluing divides the inside of the torus into two separated chambers. Compute $\pi_1(X)$.

Question 6. *Show that $\mathbb{CP}^1$ is homeomorphic to S^2.*

Question 7. *Consider $\mathbb{D}^n$ with the equivalence relation $\sim$ generated by the relation: x is equivalent to y if $x, y \in \partial\mathbb{D}^n$ and $x = \pm y$.*

(i) Show that $\mathbb{D}^n/\sim$ is homeomorphic to $\mathbb{RP}^n$.

(ii) Show that $\mathbb{RP}^1$ is homeomorphic to S^1.

Question 8. *Let $K_n = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ denote the connected sum of n-copies of $\mathbb{RP}^2$ for $n \geq 1$. Show that*

$$\pi_1(K_n) = \langle a_1, \ldots, a_n \mid a_1^2 \cdots a_n^2 \rangle.$$

Question 9. *Let $\Sigma_g = T^2 \# \cdots \# T^2 \# S^2$ denote the connected sum of g-copies of T^2 with the sphere S^2 for $g \geq 0$. Let $K_n = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ denote the connected sum of n-copies of $\mathbb{RP}^2$ for $n \geq 1$.*

(i) Show that $\Sigma_g \cong \Sigma_h$ if and only if $g = h$.

(ii) Show that $K_n \cong K_m$ if and only if $n = m$.

(iii) Show that Σ_g and K_n are not homeomorphic for all g and n.

Question 10. *Consider the simplicial complex Δ^n that is the geometric realization of the abstract simplicial complex given by $\mathcal{P}(\{0, \ldots, n\})$.*

(i) Show that the subspace $\partial\Delta^n \subseteq \Delta^n$ corresponds to the geometric realization of a subcomplex of $\mathcal{P}(\{0, \ldots, n\})$.

(ii) Show that Δ^n is homeomorphic to $\mathbb{D}^n$.

(iii) Show that $\partial\Delta^n$ is homeomorphic to S^{n-1}.

Question 11. *Show that every graph $\Gamma(V, E)$ is homeomorphic to a subspace of $\mathbb{R}^3$. (Hint: Think geometric realizations and general position.)*