18.901 - Homework 11

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Question 1 (5 points)**.** *Show that if γ and δ are two paths in X from x_0 to x_1 and γ is homotopic to δ rel ∂I, then $\Phi_\gamma = \Phi_\delta$.*

Question 2 (5 points)**.** *Let X be a path-connected space. Show that $\pi_1(X)$ is abelian if and only if all change-of-basepoint isomorphisms Φ_γ only depend on the endpoints of γ.*

Question 3 (5 points)**.** *Let G be a group. Two elements $g, g' \in G$ are conjugate if and only if there exists $h \in G$ such that $g = h \cdot g' \cdot h^{-1}$. Show that conjugation gives an equivalence relation on the group G. The conjugacy classes of G are the equivalence classes of the conjugation equivalence relation.*

Question 4 (5 points)**.** *Let X be a path-connected space. We can view $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, 0) \to (X, x_0)$. Let $[S^1, X]$ denote the set of homotopy classes of maps $S^1 \to X$ with no basepoint preserving conditions. There is a map of sets*

$$\Phi\colon \pi_1(X, x_0) \longrightarrow [S^1, X].$$

(i) Show that Φ is onto.
(ii) Show that $\Phi([\alpha]) = \Phi([\beta])$ if and only if $[\alpha]$ and $[\beta]$ are conjugate in $\pi_1(X, x_0)$.
(iii) Show that Φ induces a one-to-one correspondence between $[S^1, X]$ and conjugacy classes in $\pi_1(X, x_0)$.

Question 5 (5 points)**.** *Show that every homomorphism*

$$\mathbb{Z} \cong \pi_1(S^1) \to \pi^1(S^1) \cong \mathbb{Z}$$

can be realized as an induced homomorphism $f_*\colon \pi_1(S^1) \to \pi_1(S^1)$ for some continuous map $f\colon S^1 \to S^1$.

Question 6 (5 points)**.** *Let $p\colon E \to B$ be a fibre bundle with fibre F.*
(i) If B is path-connected and F is path-connected, does it necessarily follow that E path-connected? Prove it or give a counter-example.
(ii) If E is connected, does it necessarily follow that F is connected? Prove it or give a counter-example.

Question 7 (10 points)**.** *Let $p\colon E \to B$ be a fibre bundle with fibre F. Suppose that F and B are path-connected. Let $i\colon F \to B$ denote the inclusion of a fibre.*
(i) Show that if $\pi_1(B) = 0$, then $i_*\colon \pi_1(F) \to \pi_1(E)$ is surjective.
(ii) Show that if every map $f\colon S^2 \to B$ is null-homotopic rel a basepoint, then $i_*\colon \pi_1(F) \to \pi_1(E)$ is injective.
(iii) Show that if $\pi_1(F) = 0$, then $p_*\colon \pi_1(E) \to \pi_1(B)$ is injective.

Question 8 (5 points)**.** *Consider the equivalence relation on I^2 given by $(0, y) \sim (1, 1-y)$ and $(x, y) \sim (x, y)$. Define $M = I^2/\sim$. The space M is called the* Möbius band. *Consider the map $f\colon I^2 \to S^1$ given by $f(x, y) = (\cos(2\pi x), \sin(2\pi x))$.*
(i) Show that there exists a continuous map $\overline{f}\colon M \to S^1$ such that $\overline{f} \circ q = f$.
(ii) Show that $\overline{f}\colon M \to S^1$ is a fibre bundle. What is the fibre?

Question 9 (5 points)**.** *Consider $\mathbb{C}^{n+1} \setminus 0$ with the equivalence relation $z \sim w$ if and only if there exists $\lambda \in \mathbb{C} \setminus 0$ such that $z = \lambda w$. Define $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/\sim$.*
(i) Show that the quotient map $q\colon \mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ is a fibre bundle. What is the fibre?

(ii) Show that $\mathbb{CP}^n$ is simply-connected.

Question 10 (5 points)**.** *Let*

$$SU(n) = \{A \in GL_n(\mathbb{C}) \mid \det(A) = 1,\ AA^* = A^*A = id\},$$

where A^* denotes the conjugate transpose of A. Consider the map

$$p\colon SU(n+1) \to \mathbb{C}^{n+1} \qquad p(A) = Ae_1,$$

where $e_1 \in \mathbb{C}^{n+1}$ denotes the first standard basis vector.

(i) What is the image of p?

(ii) It turns out that $p\colon SU(n+1) \to p(SU(n+1))$ is a fibre bundle. What is the fibre?

(iii) Prove that $SU(n)$ is simply-connected for all $n \geq 0$.

Question 11 (5 points)**.** *Show that there are no retractions $r\colon X \to A$ in the following cases:*

(i) $X = \mathbb{R}^3$ and A is a subspace that is homeomorphic to S^1.

(ii) $X = S^1 \times D^2$ with $A = S^1 \times \partial D^2 \cong S^1 \times S^1$.

(iii) X is the spaced obtained from gluing together two disks along a single point on each of their boundaries and A is the boundary of X.

(iv) X is the space obtained from identifying two points on the boundary of a disk and A is the boundary of X.

(v) X is the Möbius band and A is its boundary circle.

Question 12 (5 points)**.** *Let G be a group. Given $a, b \in G$, define the* commutator *of a and b by*

$$[a, b] \coloneqq aba^{-1}b^{-1}.$$

Define the commutator subgroup *of G to be the group*

$$[G, G] \coloneqq \left\{ \prod_{i=0}^{n} [a_i, b_i] \mid a_i, b_i \in G \right\}.$$

(i) Show that $[G, G]$ is a subgroup of G.

(ii) Show that $[G, G]$ is a normal subgroup of G.

(iii) Suppose that H is an abelian group. Show that if $\varphi : G \to H$ is a homomorphism, then $[G, G] \subseteq \ker(\varphi)$.

Question 13 (5 points)**.** *The* abelianization *of a group G is the quotient $G/[G, G]$. What is the abelianization of F_n?*

Question 14 (5 points)**.** *Show that F_n and F_m (the free groups on n and m letters respectively) are isomorphic if and only if $n = m$.*

Question 15 (5 points)**.** *Give a presentation for the symmetric group on 4 letters that has two generators.*