18.901 - Homework 1

Spring 2025

Please read and follow the instructions below.

- At the beginning of your assignment, clearly write (1) the number of hours that you spent working on the assignment and (2) how challenging you found the assignment on a scale of 1 - 4 with 1 = easy, 2 = doable, 3 = challenging, but doable, 4 = very difficult.
- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.
- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

Definition. *An* equivalence relation *on a set X is a subset $R \subset X \times X$ that satisfies the following.*

(i) (Reflexivity) $(x, x) \in R$ for all $x \in X$.

(ii) (Symmetry) If $(x, y) \in R$, then $(y, x) \in R$.

(iii) (Transitivity) If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

We typically use the symbol $\sim$ to denote an equivalence relation and write $x \sim y$ to denote $(x, y) \in R$. The equivalence class *of an element $x \in X$ is the subset*

$$[x] = \{y \in X \mid x \sim y\}.$$

We use $X/\sim$ to denote the set of equivalence classes of $\sim$.

Definition. *A* partition *of a set X is a collection of disjoint non-empty subsets $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of X such that $\cup_{\alpha \in \mathcal{A}} A_\alpha = X$.*

Question 1. *Let $\sim$ be an equivalence relation on a set X.*

(i) Show that if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$.

(ii) Show that the set of equivalence classes of X defines a partition of X.

Question 2. *Let $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ be a partition of X.*

(i) Show that $x \sim y$ if and only if $x, y \in A_\alpha$ for some α defines an equivalence relation on X.

(ii) State the equivalence classes of the above equivalence relation.

Question 3. *Let $\sim$ be an equivalence relation on a set X.*

(i) Show that the assignment $q \colon X \to X/\sim$ given by $q(x) = [x]$ is a surjective map.

(ii) Let $f \colon X \to Y$ be a map of sets such that if $x \sim y$, then $f(x) = f(y)$. Show that there exists a function $\overline{f} \colon X/\sim \to Y$ such that $f = \overline{f} \circ q$.

Question 4. *Let $f \colon X \to Y$ be a surjective map of sets. We define a relation on X via $x \sim y$ if and only if $f(x) = f(y)$.*

(i) Show that $\sim$ defines an equivalence relation on X.

(ii) Describe the equivalence classes of this equivalence relation in terms of the map f and the elements of Y.