18.901– Introduction to topology

Midterm 2

MIT

Instructor: Alex Pieloch

4/17/25

Name: Solutions

Student Number: ________________

- This exam contains 22 pages and 14 questions.
- This exam is out of 64 points. The distribution of points among all of the questions is shown in the table on page 2 and is also indicated next to each question.
- Do NOT write on the backs of any pages. There are additional pages at the end of the exam if you should need them to show further work. Please indicate in your solutions when we should refer to the pages at the end of the exam for more details.
- You will have 80 minutes to complete the exam.
- Good luck!

Distribution of Marks

Question	Points	Score
1	2	
2	2	
3	3	
4	3	
5	4	
6	4	
7	4	
8	4	
9	3	
10	3	
11	6	
12	8	
13	10	
14	8	
Total:	64	

1 Definitions and statements

1. (2 points) State Lebesgue's covering lemma.

Let X = cpt metric space w/ an open cover $X = \bigcup_\alpha U_\alpha$.

$\exists\ \delta > 0$ st $\forall x \in X$, $B_x(\delta) \subseteq U_\alpha$ for some α.

2. (2 points) Given a set of points $z_0, \dots, z_m \in \mathbb{R}^n$, define what it means for this set of points to be geometrically independent and what it means for this set of points to be in general position.

- The $z_0, \dots, z_m$ are geom. ind. if

$$\textstyle\sum_i \lambda_i \cdot z_i = 0 \quad \text{and} \quad \sum_i \lambda_i = 0 \implies \lambda_i = 0$$

for $\lambda_i \in \mathbb{R}$.

- The $z_0, \dots, z_m$ are in general position if any subset of order less than or equal to $n+1$ are geom. ind.

3. (3 points) Give the definition of a deformation retraction.

A <u>deformation retraction</u> of a space X onto a subspace A is a homotopy rel A, $H: X \times I \to X$, st

① $H_0 = id_X$

② $H_1 =$ retract onto A

4. (3 points) State what it means for a continuous map $f\colon X \to Y$ to satisfy the homotopy lifting property for a space Z.

$f : X \to Y$ satisfies HLP for Z if given $\tilde{g}_0 : Z \times \{0\} \to X$ and $g : Z \times I \to Y$ st $g \circ i_0 = f \circ \tilde{g}_0$, then $\exists\, \tilde{g} : Z \times I \to X$ st

① $\tilde{g}_0 = \tilde{g} \circ i_0$

② $f \circ \tilde{g} = g$

2 True/False

For Questions 5-10, state whether or not the given statement is true or false. If it is true, provide a proof of the statement. If it is false, provide either a counter-example to the statement or a disproof.

5. (4 points) For every manifold X there exists a countable number of open subsets $U_i \subseteq X$ such that $U_i \cong \mathbb{R}^n$ and $X = \cup_i U_i$.

True

X = second countable $\Rightarrow$ $\exists$ countable basis $B_1, \dots, B_n, \dots$

X is locally Euclidean $\Rightarrow$ $\forall x \in X$ $\exists$ open $U_x \ni x$ st $U_x \cong \mathbb{R}^n$.

U_x = open $\Rightarrow$ $\exists$ basic open B_x st $x \in B_x \subseteq U_x$.

But countable # of B_x's cover X $\Rightarrow$ countable # of U_x's cover X.

6. (4 points) Let X be a connected T_1 space with $|X| > 1$. The covering dimension of X is always strictly greater than zero.

True

Fix $x \neq y \in X$.

$X = T_1 \implies X = (X \setminus x) \cup (X \setminus y)$ is open cover.

If $\dim(X) = 0 \implies \exists$ refinement $\mathcal{U}_\alpha$ st

① each x meets only one $\mathcal{U}_\alpha$

② $\mathcal{U}_\alpha \subseteq (X \setminus x)$ or $\mathcal{U}_\alpha \subseteq (X \setminus y)$.

Note $\exists\, \mathcal{U}_\alpha, \mathcal{U}_\beta$ st $x \in \mathcal{U}_\alpha \subseteq X \setminus y$, $y \in \mathcal{U}_\beta \subseteq X \setminus x$.

① $\implies \mathcal{U}_\alpha \cap \mathcal{U}_\beta = \emptyset$.

So $\mathcal{U}_\alpha, \mathcal{U}_\beta$ = open + non-empty + disjoint.

$\implies X \neq$ connected, a contradition.

So $\dim(X) > 0$. $\square$

7. (4 points) The kernel of a group homomorphism is a normal subgroup.

True

Let $\varphi : G \to H$ be a grp hom.

① $\varphi(e) = e' \Rightarrow e \in \ker(\varphi)$.

② $g \in \ker(\varphi) \Rightarrow \varphi(g^{-1}) = \varphi(g)^{-1} = (e')^{-1} = e' \Rightarrow g^{-1} \in \ker(\varphi)$

③ $g, h \in \ker(\varphi) \Rightarrow \varphi(gh) = \varphi(g) \cdot \varphi(h) = e' \cdot e' = e' \Rightarrow gh \in \ker(\varphi)$

$\Rightarrow \ker(\varphi) \subseteq G$ is a subgroup.

If $g \in G$, $n \in \ker(\varphi)$, then

$$\begin{aligned}
\varphi(gng^{-1}) &= \varphi(g) \cdot \varphi(n) \cdot \varphi(g^{-1}) \\
&= \varphi(g) \cdot e' \cdot \varphi(g^{-1}) \\
&= \varphi(g) \cdot \varphi(g^{-1}) \\
&= \varphi(g) \cdot \varphi(g)^{-1} \\
&= e'
\end{aligned}$$

$\Rightarrow gng^{-1} \in \ker(\varphi)$

$\Rightarrow \ker(\varphi) =$ normal. $\square$

8. (4 points) Let $p\colon E \to B$ be a fibre bundle with simply-connected fibre and let $[\alpha] \in \pi_1(E, x_0)$. If $p_*([\alpha])$ is the unit, then $[\alpha]$ is the unit.

True

Let $H_t : I \to B$ be a htpy rel ∂I from $p \circ \alpha$ to c.

We have a lift of H_t over $A = I \times \{0\} \cup \{0\} \times I \cup \{1\} \times I$ to a map $\tilde{H} : A \to E$ by

① $\tilde{H}_0 = \alpha$

② $\tilde{H}_t(0) = x_0 = \tilde{H}_t(1)$.

Note, $\tilde{H}_0(0) = x_0 = \alpha(0) = \alpha(1) = x_0 = \tilde{H}_1(1)$

So by the Pasting Lemma, $\tilde{H}$ is continuous over A.

Using that there exists a homeo $\varphi : I^2 \to I^2$ st

$\varphi|_{I \times \{0\}}$ maps homeomorphically onto A,

HLP $\Rightarrow \exists\ \tilde{\tilde{H}}_t : I \to E$ st

① $\tilde{\tilde{H}}_0 = \alpha$.

② $\tilde{\tilde{H}}_t(0) = x_0 = \tilde{\tilde{H}}_t(1)$.

③ $p \circ \tilde{H} = H$.

③ $\Rightarrow \tilde{H}_1(s) \in p^{-1}(H_1(s)) = p^{-1}(p(x_0)) \cong F$.

F = simply-conn $\Rightarrow \exists$ htpy rel ∂I, $\tilde{G}_t : I \to F \subseteq E$, st

① $\tilde{G}_0 = \tilde{H}_1$

② $\tilde{G}_t(0) = x_0 = \tilde{G}_t(1)$

③ $\tilde{G}_1(s) = x_0$

So $\underbrace{[\alpha] = [\tilde{H}_1]}_{\text{via } \tilde{H}_t} \underbrace{= [c]}_{\text{via } \tilde{G}_t}$ $\square$

9. (3 points) If for each $[\alpha], [\beta] \in \pi_1(X, x_0)$ there exists a continuous map of the torus $f\colon S^1 \times S^1 \to X$ such that $[\alpha] = [f|_{S^1\times\{0\}}]$ and $[\beta] = [f|_{\{0\}\times S^1}]$, then $\pi_1(X, x_0)$ is abelian.

True

Let $S^1\times\{0\}$, $\{0\}\times S^1$ be parameterized by curves $\tilde{\alpha}, \tilde{\beta}$ resp.

So $f_*([\tilde{\alpha}]) = [\alpha]$, $f_*([\tilde{\beta}]) = [\beta]$.

$\pi_1(S^1\times S^1) \cong \pi_1(S^1)\times\pi_1(S^1) \cong \mathbb{Z}\times\mathbb{Z} \Rightarrow \pi_1(S^1\times S^1) =$ abelian.

$$\begin{aligned}\Rightarrow [\alpha]\cdot[\beta] &= f_*([\tilde{\alpha}])\cdot f_*([\tilde{\beta}]) \\ &= f_*([\tilde{\alpha}]\cdot[\tilde{\beta}]) \\ &= f_*([\tilde{\beta}]\cdot[\tilde{\alpha}]) \quad \pi_1(S^1\times S^1) = \text{abelian}. \\ &\;\vdots \\ &= [\beta]\cdot[\alpha].\end{aligned}$$

$\Rightarrow \pi_1(X, x_0) =$ abelian $\square$

10. (3 points) Let A be a subspace of T^n such that T^n retracts onto A. Given $a \in A$, every non-trivial element of $\pi_1(A, a)$ has infinite order.

True

T^n retracts onto $A \Rightarrow$ inclusion $i_*: \pi_1(A,a) \to \pi_1(T^n, a)$ injective

$\pi_1(T^n) = \mathbb{Z}^n \Rightarrow$ every elm in $\pi_1(T^n)$ has infinite order.

If $\exists\ [\alpha] \in \pi_1(A,a)$ st $[\alpha]^n = 0$, then $0 = i_*([\alpha]^n) = i_*([\alpha])^n$

$\Rightarrow n = 0$ or $i_*([\alpha]) = 0$

Since $i_* = $ inj $\Rightarrow [\alpha] = 0$. $\square$

3 Free response

11. (6 points) Show that $\mathbb{R}^n$ is paracompact.

First, recall that $\mathbb{R}^n$ = Haus. So STS every open cover admits a locally finite refinement.

Let $\mathbb{R}^n = \bigcup_\alpha U_\alpha$ open cover.

HB $\Rightarrow$ $\overline{B_0(n)}$ = cpt + closed.

$\overline{B_0(n)}$ cpt $\Rightarrow$ $\exists\ U^n_{i_1}, \dots, U^n_{i_n}$ opens that cover $\overline{B_0(n)}$ w/ $U^n_{i_j} = U_\alpha$ for some α.

Define $V^n_{i_j} = U^n_{i_j} \setminus \overline{B_0(n-1)}$ = open $\setminus$ closed = open.

Claim: $V^n_{i_j}$ $\forall\, i_j, n$ give locally finite refinement

pf: ① Spse $x \in \mathbb{R}^n \Rightarrow x \in \overline{B_0(n)} \setminus B_0(n-1)$ for some n

$\Rightarrow x \in \overline{B_0(n)} \setminus B_0(n-1) \subseteq \bigcup_j V^n_{i_j}$

$\Rightarrow V^n_{i_j}$ cover $\mathbb{R}^n$.

② $V^n_{i_j} \subseteq U^n_{i_j} = U_\alpha$ for some α $\Rightarrow V^n_{i_j}$ refine original cover.

③ Spse $x \in \mathbb{R}^n \Rightarrow x \in B_0(n)$ for some n.

But $B_0(n)$ only meets $V^m_{i_j}$ for $m < n$.

$\Rightarrow V^n_{i_j}$ = locally finite. $\square$

12. (8 points) Let X be a compact Hausdorff space. Show that every open cover of X admits a refinement that is a partition of unity. (You should not use any of the results that we proved in class about partitions of unity).

Let $X = \bigcup_\alpha \mathcal{U}_\alpha$ be an open cover.

X = Cpt + Haus $\Rightarrow$ X = normal

$\Rightarrow \forall x \in \mathcal{U}_\alpha$, $\exists$ open $V_x \ni x$ st $x \in V_x \subseteq \overline{V_x} \subseteq \mathcal{U}_\alpha$

$X = \bigcup_x V_x$ + X cpt $\Rightarrow$ $\exists$ finite # $V_{x_1}, \dots, V_{x_n}$ that cover X

Spse $V_{x_i} \subseteq \mathcal{U}_{\alpha_i}$ for some α_i.

X = normal and $\overline{V}_{x_i} \cap (X \setminus \mathcal{U}_{\alpha_i}) = \phi \Rightarrow \exists$ cts $\psi_i : X \to I$ st

① $\psi_i|_{\overline{V}_{x_i}} \equiv 1$

② $\psi_i|_{X \setminus \mathcal{U}_{\alpha_i}} \equiv 0$

Define $\rho_i = \psi_i / \Sigma_j \psi_j$.

Note, V_{x_i} cover $X \Rightarrow \forall x \in X$, $\exists$ i st $\psi_i(x) \neq 0 \Rightarrow$ denominator of ρ_i is non-zero. So ρ_i is well-defined.

ρ_i is cts since it is a sum/prod. of cts fcns.

Claim: $\{(\mathcal{U}_{\alpha_i}, \rho_i)\}$ defines a partition of unity subordinate to $\mathcal{U}_\alpha$.

Pf: • $X = \bigcup_i V_{x_i} \subseteq \bigcup_i \mathcal{U}_{\alpha_i} \Rightarrow \mathcal{U}_{\alpha_i}$ = cover

• $\mathcal{U}_{\alpha_i} \subseteq \mathcal{U}_{\alpha_i} \Rightarrow \mathcal{U}_{\alpha_i}$ = refinement of orignal cover.

• Since # of $i < \infty \Rightarrow \mathcal{U}_{\alpha_i}$ is locally finite cover.

• $\rho_i(x) > 0 \Rightarrow x \notin X \setminus \mathcal{U}_{\alpha_i} \Rightarrow x \in \mathcal{U}_{\alpha_i}$

• $\Sigma_i \rho_i = \Sigma_i \rho_i / \Sigma_j \rho_j \equiv 1$. $\square$

13. Let X be a space and define

$$\text{Homeo}(X) = \{f : X \to X \mid f \text{ is a homeomorphism}\}.$$

Define an equivalence relation on Homeo(X) by $f \sim g$ if and only if f is homotopic to g. Let

$$\pi_0(\text{Homeo}(X)) = \text{Homeo}(X)/\sim.$$

(For the following questions, you should not assume any results from the class.)

(a) (4 points) Show that $\sim$ is an equivalence relation on Homeo(X).

(Reflexive): $f \in \text{Homeo}(X)$,

$H_t : X \to X$ by $H_t(x) = f \circ pr_X(x,t) = f(x)$

H_t = comp. of cts = cts

$H_0 = f$ and $H_1 = f \implies f \sim f$

(Symmetric): $f, g \in \text{Homeo}(X)$ w/ $f \sim g$.

$\implies \exists\, H_t : X \to X$ st $H_0 = f$, $H_1 = g$

Defn $G_t : X \to X$ by $G_t(x) = H_{1-t}(x) = H \circ (id \times (-(\cdot 1)))$.

G_t = comp. of cts = cts

$G_0 = H_1 = g$, $G_1 = H_0 = f \implies g \sim f$

(transitive): $f, g, h \in \text{Homeo}(X)$ w/ $f \sim g$ and $g \sim h$.

$\implies \exists\, H_t : X \to X$, $G_t : X \to X$ st

$H_0 = f$, $H_1 = g = G_0$, $G_1 = h$.

Defn

$$F_t = \begin{cases} H_{2t} & , 0 \le t \le \frac{1}{2} \\ G_{2t-1} & , \frac{1}{2} \le t \le 1 \end{cases}$$

Note $H_1 = G_0 \implies F_t$ is cts by the Pasting Lemma.

$F_0 = H_0 = f$, $F_1 = G_1 = h \implies f \sim h$. $\square$

(b) (6 points) Define

$$\bullet\colon \pi_0(\text{Homeo}(X)) \times \pi_0(\text{Homeo}(X)) \to \pi_0(\text{Homeo}(X))$$

by

$$[f] \bullet [g] = [g \circ f].$$

Show that $(\pi_0(\text{Homeo}(X)), \bullet)$ defines a group.

Claim 1: $f, g, h \in$ Homeo and $f \sim g \Rightarrow f \circ h \sim g \circ h$.

Pf: $\exists\, H_t : X \to X$ st $H_0 = f$, $H_1 = g$

Defn $G_t : X \to X$ by $G_t(x) = H_t(h(x)) = H \circ (\mathrm{id}_I \times h)(t, x)$ = comp cts = cts

$G_0 = H_0 \circ h = f \circ h$, $G_1 = H_1 \circ h = g \circ h$

$\Rightarrow f \circ h \sim f \circ g$

Claim 2: $f, g, h \in \text{Homeo}(X)$ and $f \sim g \Rightarrow h \circ f \sim h \circ g$.

Pf: $\exists\, H_t : X \to X$ st $H_0 = f$, $H_1 = g$

Defn $G_t : X \to X$ by $G_t(x) = h \circ H_t(x)$ = comp cts = cts

$G_0 = h \circ H_0 = h \circ f$, $G_1 = h \circ H_1 = h \circ g$

$\Rightarrow h \circ f \sim h \circ g$

($\bullet$ is well-defn): Spse $f \sim f'$ and $g \sim g'$.

$$[f] \cdot [g] = [g \circ f] \overset{①}{=} [g \circ f'] \overset{②}{=} [g' \circ f'] = [f'] \cdot [g']$$

(Unital): unit = id_X.

$$[\mathrm{id}_X] \cdot [f] = [f \circ \mathrm{id}_X] = [f] = [\mathrm{id}_X \circ f] = [f] \cdot [\mathrm{id}_X]$$

(associative): $[f] \cdot ([g] \cdot [h]) = [f] \cdot [h \circ g] = [(h \circ g) \circ f] \overset{(*)}{=} \ldots$

$\ldots [h \circ (g \circ f)] = [g \circ f] \cdot [h] = ([f] \cdot [g]) \cdot [h]$

$(*)$ = fcn composition is associative.

(inverses): $[f] \in \pi_0(\text{Homeo}(X)) \Rightarrow f$ = homeo $\Rightarrow f^{-1}$ = homeo

$\Rightarrow [f^{-1}] \in \pi_0(\text{Homeo}(X))$

$\Rightarrow [f] \cdot [f^{-1}] = [f^{-1} \circ f] = [\mathrm{id}_X] = [f \circ f^{-1}] = [f^{-1}] \cdot [f]$ $\square$

14. (8 points) Consider

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

and let $\mu = \exp(2\pi i/n)$, where $n \in \mathbb{Z}_{>0}$. There is an equivalence relation $\sim$ on S^3 given by $(z, w) \sim (z', w')$ if and only if $(\mu^j \cdot z, \mu^j \cdot w) = (z', w')$ for some $j \in \mathbb{Z}$. Let $L_n = S^3/\sim$ denote the quotient and let $q \colon S^3 \to L_n$ denote the quotient map. Using that q is a fibre bundle, compute the fundamental group of L_n.

Define $\tilde{\alpha} : I \to S^3$ by $\tilde{\alpha}(s) = (\exp(2\pi i \cdot s/n), 0)$.

$\tilde{\alpha}(0) \sim \tilde{\alpha}(1) \Rightarrow \alpha := q \circ \tilde{\alpha}$ defines a class in $\pi_1(L_n, [1:0])$.

Claim: $[\alpha] \neq [c]$.

Pf: Spse $\alpha \simeq c$ rel ∂I via a htpy $H_t : I \to L_n$.

We have a lift of H_t over $A = I \times \{0\} \cup \{0\} \times I \cup \{1\} \times I$ to a map $\tilde{H} : A \to E$ by

① $\tilde{H}_t(0) = (1,0) = \tilde{H}_1(s)$

② $\tilde{H}_0(s) = \tilde{\alpha}(s)$

Note, $\tilde{H}_0(0) = x_0 = \tilde{\alpha}(0)$

So by the Pasting Lemma, $\tilde{H}$ is continuous over A.

Using that there exists a homeo $\varphi : I^2 \to I^2$ st $\varphi|_{I \times \{0\}}$ maps homeomorphically onto A

By HLP, $\exists$ htpy $\tilde{H}_t : I \to S^3$ st

① $\tilde{H}_t(0) = (1,0) = \tilde{H}_1(s)$

② $\tilde{H}_0(s) = \tilde{\alpha}(s)$

③ $q \circ \tilde{H}_t(1) = [1:0]$.

③ $\Rightarrow \tilde{H}_t(1) \subseteq q^{-1}([1:0]) = \{(\mu^1, 0), \ldots, (\mu^n, 0)\}$ = discrete.

$\tilde{H}_t(1)$ = cts map w/ $\tilde{H}_0(1) = \tilde{\alpha}(1) = (\mu, 0)$, $\tilde{H}_1(1) = (1,0)$.

Since $\tilde{H}_{(I)}(1)$ = connected $\Rightarrow \mu = 1 \Rightarrow \Leftarrow$. $\square$

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Claim: $[\gamma] \in \pi_1(L_n, [1:0]) \Rightarrow [\gamma] = [\alpha]^k$ for some k.

Pf: HLP $\Rightarrow \exists\ \tilde{\gamma} : I \to S^3$ st $\tilde{\gamma}(0) = (1,0)$, $q \circ \tilde{\gamma} = \gamma$.

$\tilde{\gamma}(1) \in q^{-1}(\gamma(1)) \Rightarrow \tilde{\gamma}(1) = (\mu^k, 0)$.

S^3 = simply-conn $\Rightarrow \exists$ htpy $\tilde{H}_t : I \to S^3$ rel ∂I from $\tilde{\gamma}$ to $\tilde{\alpha}^k$.

$q \circ \tilde{H}_t(0) = [1:0] = q \circ \tilde{H}_t(1)$, $q \circ \tilde{H}_0 = \gamma$, $q \circ \tilde{H}_1 = \alpha^k$.

$\Rightarrow q \circ \tilde{H}$ = htpy rel ∂I from γ to α^k. $\square$

Note, the above shows that $\alpha^n \simeq \alpha^0 = c$ rel ∂I.

$\Rightarrow \pi_1(L_n, [1:0]) = \{\alpha^0, \dots, \alpha^{n-1}\} = \mathbb{Z}/n\mathbb{Z}$ $\square$

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