18.901– Introduction to topology

Midterm 2

MIT

Instructor: Alex Pieloch

4/17/25

Name: ______________________________

Student Number: ______________________________

- This exam contains 22 pages and 14 questions.
- This exam is out of 64 points. The distribution of points among all of the questions is shown in the table on page 2 and is also indicated next to each question.
- Do NOT write on the backs of any pages. There are additional pages at the end of the exam if you should need them to show further work. Please indicate in your solutions when we should refer to the pages at the end of the exam for more details.
- You will have 80 minutes to complete the exam.
- Good luck!

Distribution of Marks

Question	Points	Score
1	2	
2	2	
3	3	
4	3	
5	4	
6	4	
7	4	
8	4	
9	3	
10	3	
11	6	
12	8	
13	10	
14	8	
Total:	64	

1 Definitions and statements

1. (2 points) State Lebesgue's covering lemma.

2. (2 points) Given a set of points $z_0, \dots, z_m \in \mathbb{R}^n$, define what it means for this set of points to be geometrically independent and what it means for this set of points to be in general position.

3. (3 points) Give the definition of a deformation retraction.

4. (3 points) State what it means for a continuous map $f\colon X \to Y$ to satisfy the homotopy lifting property for a space Z.

2 True/False

For Questions 5-10, state whether or not the given statement is true or false. If it is true, provide a proof of the statement. If it is false, provide either a counter-example to the statement or a disproof.

5. (4 points) For every manifold X there exists a countable number of open subsets $U_i \subseteq X$ such that $U_i \cong \mathbb{R}^n$ and $X = \cup_i U_i$.

6. (4 points) Let X be a connected T_1 space with $|X| > 1$. The covering dimension of X is always strictly greater than zero.

7. (4 points) The kernel of a group homomorphism is a normal subgroup.

8. (4 points) Let $p\colon E \to B$ be a fibre bundle with simply-connected fibre and let $[\alpha] \in \pi_1(E, x_0)$. If $p_*([\alpha])$ is the unit, then $[\alpha]$ is the unit.

9. (3 points) If for each $[\alpha], [\beta] \in \pi_1(X, x_0)$ there exists a continuous map of the torus $f\colon S^1 \times S^1 \to X$ such that $[\alpha] = [f|_{S^1 \times \{0\}}]$ and $[\beta] = [f|_{\{0\} \times S^1}]$, then $\pi_1(X, x_0)$ is abelian.

10. (3 points) Let A be a subspace of T^n such that T^n retracts onto A. Given $a \in A$, every non-trivial element of $\pi_1(A, a)$ has infinite order.

3 Free response

11. (6 points) Show that $\mathbb{R}^n$ is paracompact.

12. (8 points) Let X be a compact Hausdorff space. Show that every open cover of X admits a refinement that is a partition of unity. (You should not use any of the results that we proved in class about partitions of unity).

13. Let X be a space and define

$$\mathrm{Homeo}(X) = \{f\colon X \to X \mid f \text{ is a homeomorphism}\}.$$

Define an equivalence relation on $\mathrm{Homeo}(X)$ by $f \sim g$ if and only if f is homotopic to g. Let

$$\pi_0(\mathrm{Homeo}(X)) = \mathrm{Homeo}(X)/\sim .$$

(For the following questions, you should not assume any results from the class.)

(a) (4 points) Show that $\sim$ is an equivalence relation on $\mathrm{Homeo}(X)$.

(b) (6 points) Define

$$\bullet \colon \pi_0(\mathrm{Homeo}(X)) \times \pi_0(\mathrm{Homeo}(X)) \to \pi_0(\mathrm{Homeo}(X))$$

by

$$[f] \bullet [g] = [g \circ f].$$

Show that $(\pi_0(\mathrm{Homeo}(X)), \bullet)$ defines a group.

14. (8 points) Consider

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

and let $\mu = \exp(2\pi i/n)$, where $n \in \mathbb{Z}_{>0}$. There is an equivalence relation $\sim$ on S^3 given by $(z, w) \sim (z', w')$ if and only if $(\mu^j \cdot z, \mu^j \cdot w) = (z', w')$ for some $j \in \mathbb{Z}$. Let $L_n = S^3/\sim$ denote the quotient and let $q\colon S^3 \to L_n$ denote the quotient map. Using that q is a fibre bundle, compute the fundamental group of L_n.

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